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MATHEMATICS MAGAZINE

Clavius and the Food Chain

- A Lotka-Volterra Three-species Food Chain
- Teaching Mathematics in the Seventeenth and Twenty-first Centuries
- Counting Trash in Poker

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Cover image: Clavius and the Food Chain, by Jason Challas. While fish eats fish eats fish, it seems that Clavius's hand contains nothing better than trash. Jason Challas lectures on computer art and the fish-eat-fish world at Santa Clara University.

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A Lotka-Volterra Three-species Food Chain

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In the 1920s, the Italian mathematician Vito Volterra [**8**] proposed a differential equation model to describe the population dynamics of two interacting species, a predator and its prey. He hoped to explain the observed increase in predator fish (and corresponding decrease in prey fish) in the Adriatic Sea during World War I. Such mathematical models have long proven useful in describing how populations vary over time. Data about the various rates of growth, death, and interaction of species naturally lead to models involving differential equations.

Independently, in the United States, the very equations studied by Volterra were derived by Alfred Lotka [**6**] to describe a hypothetical chemical reaction in which the chemical concentrations oscillate.

The Lotka-Volterra model [**2**] consists of the following system of differential equations:

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy, \\ \dfrac{dy}{dt} = -cy + dxy, \end{cases} \tag{1}$$

where $y(t)$ and $x(t)$ represent, respectively, the predator population and the prey population as functions of time. The parameters $a, b, c, d > 0$ are interpreted as follows:

- a represents the natural growth rate of the prey in the absence of predators,
- b represents the effect of predation on the prey,
- c represents the natural death rate of the predator in the absence of prey,
- d represents the efficiency and propagation rate of the predator in the presence of prey. (Using the letter d is traditional; the reader is trusted to observe from context when we wish to indicate differentiation using this same letter.)

To analyze this system in a straightforward manner, divide the second equation of (1) by the first,

$$\frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = \frac{y(-c + dx)}{x(a - by)},$$

and solve the resulting separable ODE. This gives the family of equations

$$a \ln y - by + c \ln x - dx = C,$$

where C is the constant of integration. It can be shown that the maximum value C_* of the left-hand side of the above equation occurs at $(c/d, a/b)$. FIGURE 1 depicts the family of trajectories of (1) in the xy-plane with parameters $a = b = c = d = 1$; readers may test their biological savvy by determining why the trajectories circulate counterclockwise. FIGURE 2 plots the behavior of a particular solution over time.

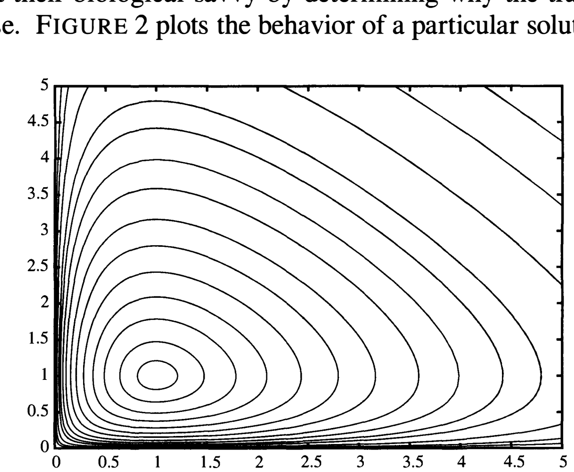

Figure 1 A family of closed orbits in the xy-plane circulating about $(1, 1)$ with $a = b = c = d = 1$

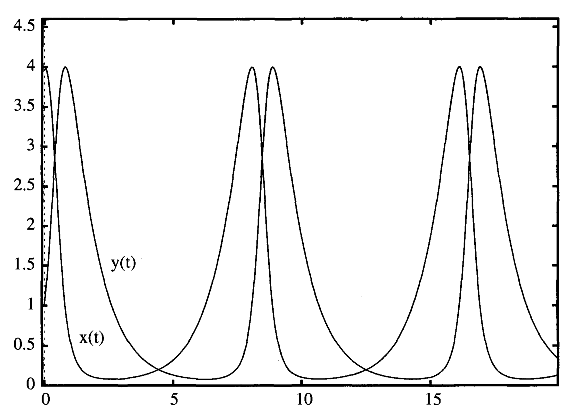

Figure 2 A solution with initial conditions $(x, y) = (4, 1)$ with parameters $a = b = c = d = 1$

The predator-prey model predicts a phase-shifted periodic behavior in the populations of both species with a common period. This behavior is seen in the historical records of the Hudson's Bay Company, which recorded the annual number of pelts of hare (prey) and lynx (predator) collected from 1845–1935 [**3**].

In this paper, we completely characterize the qualitative behavior of a linear three-species food chain where the dynamics are given by classic (nonlogistic) Lotka-Volterra type equations. The Lotka-Volterra equations are typically modified by making the prey equation a logistic (Holling-type [**5**]) equation to eliminate the possibility of unbounded growth of the prey in the absence of the predator. We study a more basic nonlogistic system that is the direct generalization of the classic Lotka-Volterra equations. Although the model is more simplified, the dynamics of the associated system are quite complicated, as the model exhibits degeneracies that make it an excellent instructional tool whose analysis involves advanced topics such as: trapping regions, nonlinear analysis, invariant sets, Lyapunov-type functions (F and G in what follows), the stable/center manifold theorem, and the Poincaré-Bendixson theorem.

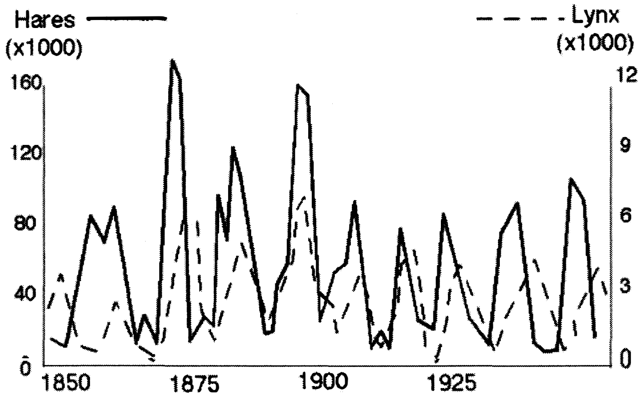

Figure 3 Historical plots of hare and lynx pelts collected by the Hudson's Bay Company

The model

The ecosystem that we wish to model is a linear three-species food chain where the lowest-level prey x is preyed upon by a mid-level species y, which, in turn, is preyed upon by a top level predator z. Examples of such three-species ecosystems include: mouse-snake-owl, vegetation-hare-lynx, and worm-robin-falcon. The model we propose to study is

$$\begin{cases} \dfrac{dx}{dt} = ax - bxy \\ \dfrac{dy}{dt} = -cy + dxy - eyz \\ \dfrac{dz}{dt} = -fz + gyz, \end{cases} \tag{2}$$

for $a, b, c, d, e, f, g > 0$, where a, b, c and d are as in the Lotka-Volterra equations and:

- e represents the effect of predation on species y by species z,
- f represents the natural death rate of the predator z in the absence of prey,
- g represents the efficiency and propagation rate of the predator z in the presence of prey.

Since populations are nonnegative, we will restrict our attention to the nonnegative octant $\{(x, y, z) \mid x \geq 0, y \geq 0, z \geq 0\} \subset \mathbb{R}^3$ and the positive octant $\mathbb{R}^3_+ = \{(x, y, z) \mid x > 0, y > 0, z > 0\} \subset \mathbb{R}^3$.

Analysis of the model

The coordinate planes We first show that each coordinate plane is invariant with respect to the system (2). In general, a surface S is invariant with respect to a system of differential equations if every solution that starts on S does not escape S. The property of invariant coordinate planes matches biological considerations, since if some species is extinct, it will not reappear.

The following result appears in most advanced texts on differential equations [**1**]. In such texts, the invariant surface S is usually given as the level set of a function $G(x, y, z)$, which is called a first integral of the system (2).

THEOREM 1. *Let S be a smooth closed surface without boundary in $\mathbb{R}^3$ and*

$$\begin{aligned}\frac{dx}{dt} &= f(x, y, z),\\ \frac{dy}{dt} &= g(x, y, z),\\ \frac{dz}{dt} &= h(x, y, z),\end{aligned} \tag{3}$$

where f, g, and h are continously differentiable. Suppose that $\mathbf{n}$ is a normal vector to the surface S at (x, y, z), and for all $(x, y, z) \in S$ we have that

$$\mathbf{n} \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = 0.$$

Then S is invariant with respect to the system (3).

Let S be the plane $z = 0$, note that the vector $\langle 0, 0, 1\rangle$ is always normal to S, and that at the point $(x, y, 0)$ of S we have

$$\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle = \langle ax - bxy, -cy + dxy, 0\rangle.$$

Thus,

$$\langle 0, 0, 1\rangle \cdot \langle ax - bxy, -cy + dxy, 0\rangle = 0.$$

Similar arguments show that each coordinate plane is invariant.

Next, we solve each of the three corresponding planar (two variable) systems in the respective coordinate planes. We first notice that in absence of the top predator ($z = 0$), the model reduces to the classic Lotka-Volterra equations with closed trajectories centered at the equilibrium $(c/d, a/b, 0)$, for all values of the parameters.

For a trajectory starting on the plane $y = 0$, equations (2) reduce to:

$$\begin{cases}\dfrac{dx}{dt} = ax\\ \dfrac{dy}{dt} = 0\\ \dfrac{dz}{dt} = -fz.\end{cases}$$

The equation $dz/dt = -fz$ implies that $z(t) \to 0$ exponentially as $t \to \infty$, while $dx/dt = ax$ implies that x grows exponentially as $t \to \infty$. Notwithstanding the unbounded growth of x, this behavior fits with what we would expect biologically in the absence of mid-level species y. That is, x is free from predation and z is without a source of food. The trajectories in the xz-plane can be directly computed from the separable equation:

$$\frac{dz}{dx} = \frac{dz}{dt} \bigg/ \frac{dx}{dt} = \frac{-fz}{ax}$$

that has solution $z = Kx^{-f/a}$ (see FIGURE 4).

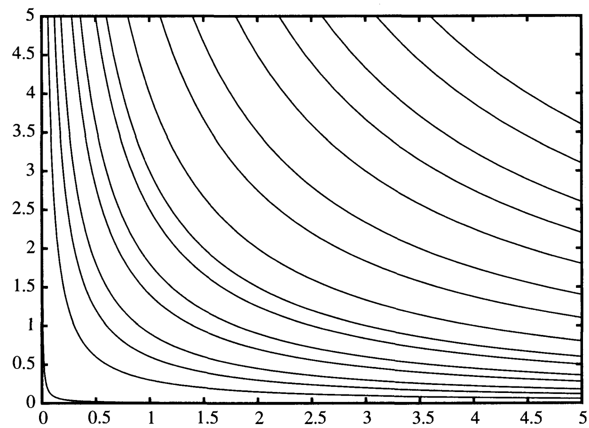

Figure 4 A family of trajectories $z = Kx^{-f/a}$ in the xz-plane with $a = b = c = d = e = f = g = 1$

For solutions starting in the plane $x = 0$, we see that (2) reduces to

$$\begin{cases} \dfrac{dx}{dt} = 0 \\ \dfrac{dy}{dt} = -cy - eyz \\ \dfrac{dz}{dt} = -fz + gyz. \end{cases}$$

Since $dy/dt \le -cy$, as $t \to \infty$ we have $y(t) \to 0$. This, in turn, will cause $z(t) \to 0$, as $t \to \infty$. Note that dz/dy is also separable and has solutions of the form

$$-f \ln y + gy = -c \ln z - ez + K$$

in the yz-plane. The reader may wish to verify that, if a solution starts in this plane with $y > f/g$, then z will have a maximum value when y has dropped back to f/g. This makes sense biologically: under these circumstances, the predator z may temporarily grow in numbers while it exhausts the prey y, but then tend to extinction itself, having no replenished food source. To summarize, all species eventually become extinct in the absence of bottom-level prey x.

Equilibria and linear analysis In the analysis of systems of differential equations it is often useful to consider solutions that do not change with time, that is, for which $dx/dt = 0$, $dy/dt = 0$, and $dz/dt = 0$. Such solutions are called equilibria, steady-states, or fixed points. For system (2) there are two equilibria located at $(0, 0, 0)$ and $(c/d, a/b, 0)$. The special case $a/b = f/g$, which we will study in detail later as a borderline case, yields a ray of fixed points parameterized by: $(s, a/b, (ds - c)/e)$, where $s \ge c/d$.

An equilibrium is called asymptotically stable if solutions starting close enough to the equilibrium tend to that equilibrium. If the system of differential equations (3) can be linearized, that is, if f, g, and h have continuous partials in x, y, and z, then the stability of an equilibrium (x_0, y_0, z_0) often can be determined by the stability of

(x_0, y_0, z_0) in the associated linearized system:

$$\begin{cases} \dfrac{dx}{dt} = \frac{\partial f}{\partial x}x + \frac{\partial f}{\partial y}y + \frac{\partial f}{\partial z}z \\ \dfrac{dy}{dt} = \frac{\partial g}{\partial x}x + \frac{\partial g}{\partial y}y + \frac{\partial g}{\partial z}z \\ \dfrac{dz}{dt} = \frac{\partial h}{\partial x}x + \frac{\partial h}{\partial y}y + \frac{\partial h}{\partial z}z, \end{cases}$$

where all partials are evaluated at (x_0, y_0, z_0). The behavior of the linearized system at (x_0, y_0, z_0) is determined by the eigenvalues of the Jacobian matrix:

$$J(x, y, z) = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} & \dfrac{\partial g}{\partial z} \\ \dfrac{\partial h}{\partial x} & \dfrac{\partial h}{\partial y} & \dfrac{\partial h}{\partial z} \end{bmatrix}$$

evaluated at (x_0, y_0, z_0).

For system (2),

$$J(x, y, z) = \begin{bmatrix} a - by & -xb & 0 \\ yd & -c + dx - ez & -ye \\ 0 & zg & -f + gy \end{bmatrix}.$$

Simply examining the eigenvalues of $J(x_0, y_0, z_0)$ gives us information about the dynamics near the equilibrium of the original system. If all eigenvalues of $J(x_0, y_0, z_0)$ have negative real part then (x_0, y_0, z_0) is asymptotically stable. If any eigenvalue has positive real part then (x_0, y_0, z_0) is not asymptotically stable.

A powerful tool used to analyze the dynamics of a nonlinear system near an equilibrium is the Center Manifold Theorem (for a technical treatment, see Chapter 3 of Guckenheimer and Holmes [**4**]). The reader unfamiliar with manifolds will find that, informally, this is a generic term to encompass sets such as nonsingular curves and surfaces.

The Center Manifold Theorem states that associated with each equilibrium (x_0, y_0, z_0) there exist invariant sets containing (x_0, y_0, z_0), called the stable manifold, the unstable manifold, and a center manifold. The stable and unstable manifolds are unique, but there may be more than one center manifold.

The dimensions of these sets are given by the number of eigenvalues of $J(x_0, y_0, z_0)$ having negative, positive, and zero real part, respectively. Moreover, each such manifold is tangent to the real space that is spanned by the eigenvectors associated with that manifold. On the stable manifold all trajectories tend toward the equilibrium as $t \to \infty$, and on the unstable manifold all trajectories tend away from the equilibrium as $t \to \infty$. However, the theorem gives no conclusion concerning the direction of the flow of trajectories on a center manifold.

For example, suppose $J(x_0, y_0, z_0)$ has a single eigenvalue λ_1 having positive real part, and two eigenvalues λ_2, λ_3 having negative real part. Then there exists an unstable manifold associated with λ_1, which is a one-dimensional curve passing through (x_0, y_0, z_0) that is tangent to the eigenvector corresponding to λ_1. There also exists an invariant stable manifold, which is a two-dimensional surface containing (x_0, y_0, z_0) whose tangent plane at (x_0, y_0, z_0) is given by the span of the eigenvectors associated with λ_2 and λ_3. In this case, there is no center manifold.

Let us apply this theorem to the equilibrium $(0, 0, 0)$ of system (2). The Jacobian matrix $J(0, 0, 0)$ has eigenvalues a, $-c$, and $-f$ with eigenvectors $\langle 1, 0, 0\rangle$, $\langle 0, 1, 0\rangle$, and $\langle 0, 0, 1\rangle$, respectively. First note that this equilibrium is not asymptotically stable since we have an eigenvalue with positive real part, namely a. By the Center Manifold Theorem, this positive eigenvalue has a corresponding one-dimensional unstable manifold (a curve), which is tangent to $\langle 1, 0, 0\rangle$ at the equilibrium $(0, 0, 0)$. As we saw previously, this unstable curve turns out to be the x-axis. Corresponding to the two negative eigenvalues is an invariant two-dimensional stable manifold, that is, an invariant surface through $(0, 0, 0)$ on which solutions move toward $(0, 0, 0)$. This stable surface intersects $(0, 0, 0)$ tangent to the plane spanned by the two eigenvectors $\langle 0, 1, 0\rangle$ and $\langle 0, 0, 1\rangle$. Again, using our previous analysis, we see that the stable manifold turns out to be the yz-plane itself. Note that as there are no eigenvalues with zero real part, so there is no center manifold in this case.

At the equilibrium $(c/d, a/b, 0)$, the eigenvalues of $J(c/d, a/b, 0)$ are $(ga - fb)/b$ along with the purely imaginary numbers $\pm\sqrt{ac}\,i$. For $ga \neq fb$ we use the Center Manifold Theorem to conclude that associated with the eigenvalue $(ga - fb)/b$ there is a one-dimensional invariant curve, which is tangent at $(c/d, a/b, 0)$ to the eigenvector $\langle 1, (fb - ag)d/b^2c, (ab^2cd + b^2df^2 - 2abdfg + a^2dg^2)/ab^2ce\rangle$. This curve is stable if $ga - fb < 0$ and unstable if $ga - fb > 0$. Corresponding to the two eigenvalues with zero real part, there exists a two-dimensional invariant center manifold passing through $(c/d, a/b, 0)$ tangent to the two-dimensional real subspace of the complex space spanned by the complex eigenvectors $\langle 1, \pm(\sqrt{acd})/bc\,i, 0\rangle$, namely the xy-plane itself. Using our prior analysis, we can conclude that the xy-plane is indeed invariant, and hence is a center manifold, but we cannot conclude that it is unique.

In the special case $ga = fb$, the Jacobian matrix evaluated at any of the fixed points $(s, a/b, (ds - c)/e)$, with $s \geq c/d$, has three eigenvalues with zero real part. Thus, each such fixed point has a three-dimensional center manifold, which does not help us determine the dynamics near these fixed points.

The case $ga = fb$ Plots of trajectories using a numerical solver suggest that the system contains invariant surfaces. In particular, the projection of any trajectory in $\mathbb{R}^3_+$ onto the xz-plane lies entirely within one of the curves depicted in FIGURE 4. This suggests that the surfaces $z = Kx^{-f/a}$ are invariant in $\mathbb{R}^3_+$. These surfaces are filled with periodic orbits enclosing the ray of fixed points $(s, a/b, (ds - c)/e)$, with $s \geq c/d$ (see FIGURE 5). We will demonstrate this analytically later.

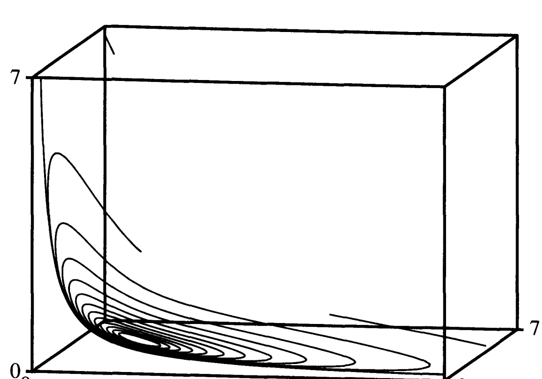

Figure 5 A family of closed orbits on the surface $z = Kx^{-f/a}$ in xyz-space with parameters $a = b = c = d = e = f = g = K = 1$

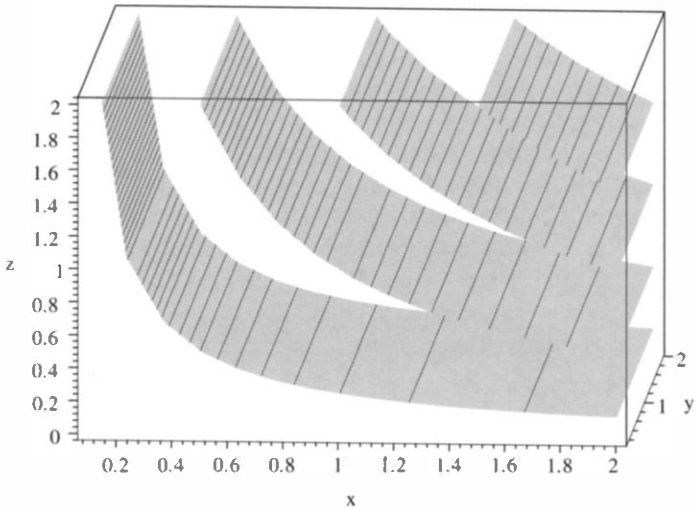

Figure 6 The surfaces $z = Kx^{-f/a}$ with $a = f = 1$

PROPOSITION 2. *Let $ga = fb$. The surfaces defined by $z = Kx^{-f/a}$ are invariant with respect to (2).*

Proof. The vector $\mathbf{n} = \langle (Kf/a)x^{-(1+f/a)}, 0, 1\rangle$ is always normal to $-Kx^{-f/a} + z = 0$. Consider

$$\begin{aligned}\mathbf{n}\cdot\left\langle\frac{dx}{dt},\frac{dy}{dt},\frac{dz}{dt}\right\rangle\\ &= \left\langle K\frac{f}{a}x^{-\frac{f}{a}-1}, 0, 1\right\rangle\cdot\langle ax - bxy, -cy + dxy - eyz, -fz + gyz\rangle\\ &= (ax - bxy)\left(K\frac{f}{a}x^{-\frac{f}{a}-1}\right) - fz + gyz\\ &= fKx^{-\frac{f}{a}} - \frac{bfyK}{a}x^{-\frac{f}{a}} - fKx^{-\frac{f}{a}} + gyKx^{-\frac{f}{a}} = 0.\end{aligned}$$

Thus, the surface $z = Kx^{-f/a}$ is invariant with respect to (2). ∎

Next, we implicitly solve the differential equation on each surface. For fixed K with $z = Kx^{-f/a}$, the system (2) becomes

$$\begin{cases}\dfrac{dx}{dt} = ax - bxy,\\ \dfrac{dy}{dt} = -cy + dxy - eyKx^{-\frac{f}{a}}.\end{cases} \tag{4}$$

Again taking the quotient of the equations, we find a separable equation,

$$\frac{dy}{dx} = \frac{dy}{dt}\Big/\frac{dx}{dt} = \frac{y(-c + dx - eKx^{-\frac{f}{a}})}{x(a - by)},$$

which we solve to obtain the implicit solution:

$$a\ln y - by + c\ln x - dx - \frac{eaK}{f}x^{-\frac{f}{a}} = C \tag{5}$$

on the surface $z = Kx^{-f/a}$. For fixed K, the family of closed curves (5) $C \leq C_K$, fills up the surface $z = Kx^{-f/a}$, where C_K is the maximum value of the left-hand side of (5). This value is attained at the fixed point given by the intersection of the line $(s, a/b, (ds - c)/e)$ and the surface $z = Kx^{-f/a}$.

We sketch the proof that the trajectory given by (5) is closed for any fixed $C < C_K$. One first uses techniques from calculus to show that the trajectory lies in a bounded region. Since $C < C_K$ is fixed, this trajectory cannot tend to the fixed point associated with C_K. The Poincaré-Bendixson Theorem [**4**] tells us that the trajectory is either a closed orbit or tends to a closed orbit. To rule out the latter, one can show that for fixed x, the solutions of (5) have at most two solutions for y, and vice versa.

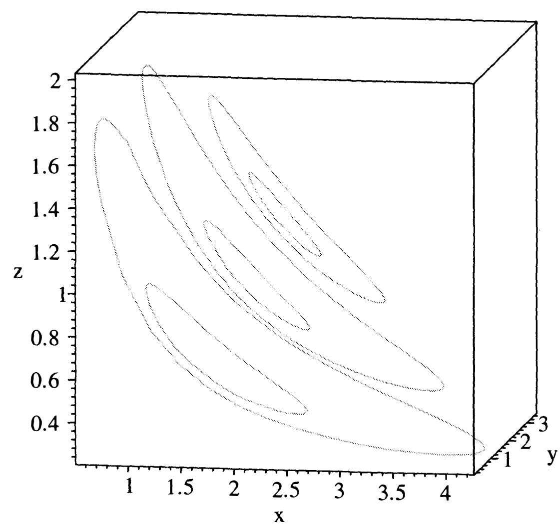

Figure 7 Several closed trajectories with $a = b = c = d = e = f = g = 1$

This completely characterizes the behavior in the special case $ga = fb$. Biologically, all three species persist and have populations that vary periodically over time with a common period. FIGURE 8 below depicts the plot of a particular periodic solution. Note the relative positions of the three maxima, with the lowest-level prey population peaking first, followed by a peak in the population in the mid-level species, with

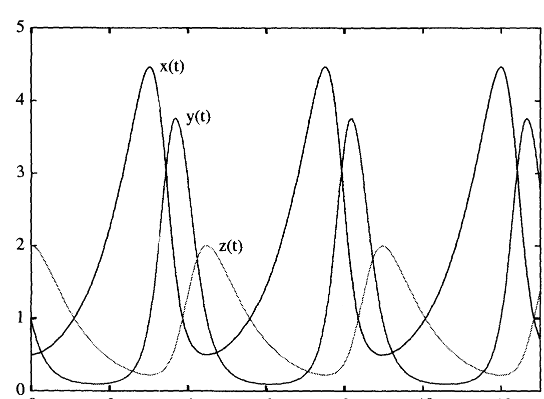

Figure 8 A solution with initial conditions $(x, y, z) = (.5, 1, 2)$ with parameters $a = b = c = d = e = f = g = 1$

the maximum in the top predator population coming last, as one would expect from biological considerations.

The cases $ga \neq fb$ We first consider the case $ga < fb$. Plots of solutions using a numerical solver suggest that all solutions spiral down to the xy-plane and tend to a periodic solution (see FIGURE 9).

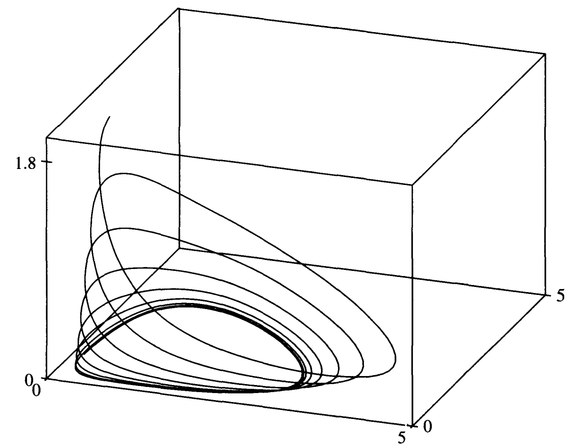

Figure 9 A trajectory in xyz-space with initial conditions $(\frac{1}{2}, 1, 2)$ with $a = b = c = d = e = f = 1$ and $g = 0.88$

We show that solutions move down across the surfaces $z = Kx^{-f/a}$ from higher values of K to lower values of K. More formally:

PROPOSITION 3. *Let $ga < fb$ and $F(x, y, z) = zx^{f/a}$. Then for any solution $(x(t), y(t), z(t))$ of (1) in $\mathbb{R}^3_+$ we have*

$$\frac{d}{dt}F(x(t), y(t), z(t)) < 0.$$

Proof.

$$\begin{aligned}\frac{d}{dt}F(x(t), y(t), z(t)) &= \nabla F(x(t), y(t), z(t)) \cdot \langle x'(t), y'(t), z'(t)\rangle \\ &= (ax - bxy)\frac{f}{a}zx^{\frac{f}{a}-1} + x^{\frac{f}{a}}(-fz + gyz) \\ &= \left(g - \frac{bf}{a}\right)yzx^{\frac{f}{a}} < 0.\end{aligned}$$ ■

This proposition implies that solutions travel down across the level surfaces of the function F, which are precisely $K = zx^{f/a}$. This proposition, however, is not sufficient to conclude that all solutions approach the plane $z = 0$, since the surfaces $z = Kx^{-f/a}$ tend to the union of the coordinate planes $z = 0$ and $x = 0$ as $K \to 0$.

To show that all solutions approach $z = 0$, we build a compact trapping region consisting of the xy-plane and the surface $by - a\ln y + dx - c\ln x + aez/f = C$. This surface arises naturally by extending equation (5) to three dimensions. Intuitively, this previously invariant surface (in the case $ga = fb$) will no longer be invariant in the case $ga < fb$, but will aid us in the construction of a trapping region.

Such a trapping region is bounded by the surface shown in FIGURE 10.

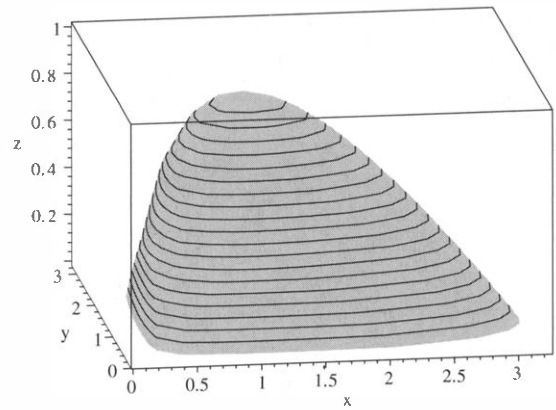

Figure 10 A particular trapping region with $a = b = c = d = e = f = 1$

PROPOSITION 4. *Let $ga < fb$ and $G(x, y, z) = by - a\ln y + dx - c\ln x + aez/f$. Then for any solution $(x(t), y(t), z(t))$ of (1) in $\mathbb{R}^3_+$ we have*

$$(d/dt)G(x(t), y(t), z(t)) < 0.$$

Proof. Let $(x(t), y(t), z(t))$ be a solution to (2) in $\mathbb{R}^3_+$. Consider

$$\begin{aligned}\frac{d}{dt}[G(x(t), y(t), z(t))] &= \nabla\mathbf{G}(x, y, z) \cdot \langle x'(t), y'(t), z'(t)\rangle \\ &= (d - c/x)(a - bxy) + (b - a/y)(-cy + dxy - eyz) \\ &\quad + (ae/f)(-fz + gyz) \\ &= eyz(ag/f - b) < 0.\end{aligned}$$ ∎

The above proposition implies that solutions travel down the level surfaces of G as time increases. In particular, a solution starting with initial condition (x_0, y_0, z_0) at time t_0 can never travel to a region in $\mathbb{R}^3_+$ where $G(x, y, z) \geq G(x_0, y_0, z_0)$. Further, since the xy-plane is invariant, the solution will be trapped in the region bounded below by the xy-plane and above by the surface $by - a\ln y + dx - c\ln x + aez/f = G(x_0, y_0, z_0)$ for all $t > t_0$ (see FIGURE 10).

Taking both propositions together, we have shown that for $ga < fb$, all trajectories beginning in $\mathbb{R}^3_+$ tend to the plane $z = 0$. Some further analysis can be done to show that the center manifold of the fixed point $(c/d, a/b, 0)$ is the plane $z = 0$ and that it is unique. Thus, except for the one trajectory that corresponds to the stable manifold of $(c/d, a/b, 0)$, all solutions approach a periodic solution in the xy-plane. Biologically, this implies that the top predator z tends to extinction, while the species x and y tend to exhibit traditional Lotka-Volterra periodic behavior in the absence of z. The plot of the solution in FIGURE 11 exhibits this behavior.

The case $ga > fb$ is analogous, except that all trajectories starting in $\mathbb{R}^3_+$ escape each of the previous trapping regions and all trajectories travel up the surfaces $z = Kx^{-f/a}$, that is, $z(t)$ approaches $+\infty$ as $t \to \infty$. This implies that the populations of the species x and z tend to $+\infty$, albeit non-monotonically, while the population of y over time experiences larger and larger fluctuations. FIGURE 12 depicts a particular solution that exhibits this behavior.

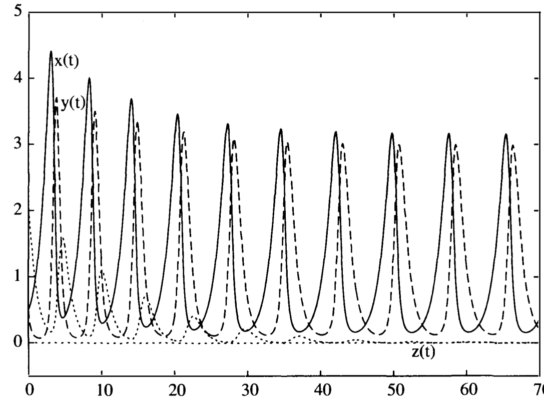

Figure 11 A solution with initial conditions (1/2), (1/2), 2) with $a = b = c = d = e = f = 1$ and $g = 0.88$

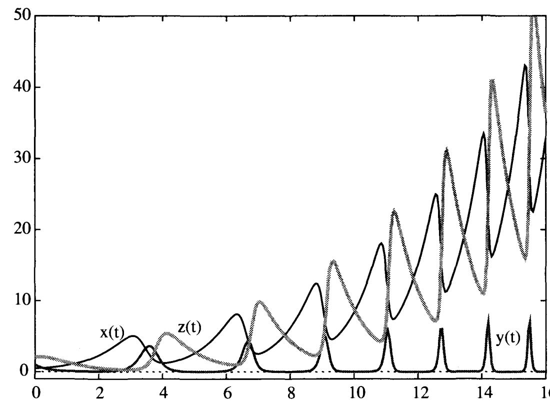

Figure 12 A solution with initial conditions $(x, y, z) = (.5, 1, 2)$ with parameters $g = 1.6$ and $a = b = c = d = e = f = 1$

Conclusion and comments

The overall long-term persistence of top species z in (2) hinges solely on the parameters a, b, f, and g. In particular, if $ag < fb$, then species z dies out, while if $ag \geq fb$, then species z survives, growing without bound in the case $ag > fb$. This coincides with our intuition as larger values of a and g are explicitly beneficial to species z, while larger values of b and f are inhibitory to species z. It is quite interesting to note that the parameters most directly related to mid-level species y (namely c, d and e) in no way affect whether species z will become extinct or not. In effect, species y is simply acting as a conduit between the top and bottom species. Furthermore, the model does not allow for the possibility of the extinction of species y while species x persists.

This basic model of a three-species food chain makes an excellent guided exploration/group project in a mathematical modeling or differential equations course and can easily be modified to model more complicated biological behavior. Lastly, the analysis of this model involves many advanced techniques typically used in the analysis of more complicated biological models that are not typically seen in standard models, e.g., non-isolated fixed points, trapping regions, and invariant sets.

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Proof by Poem: The RSA Encryption Algorithm

Take two large prime numbers, q and p.
Find the product n, and the totient ϕ.
If e and ϕ have GCD one
and d is e's inverse, then you're done!
For sending m raised to the e
reduced mod n gives secre-c.

—DANIEL G. TREAT
NATIONAL SECURITY AGENCY

Teaching Mathematics in the Seventeenth and Twenty-first Centuries

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In the late 1960s, many people saw a fictional vision of the beginning of the twenty-first century via the movie, *2001: A Space Odyssey*. Early in the movie, a lunar expedition uncovers a large, black monolith in the crater Clavius. Although the movie was fictional, and computers have not yet reached HAL's ability to speak and read lips, the lunar crater Clavius does exist and is named after a sixteenth century scholar who was instrumental in introducing mathematics into the university curriculum.

Christopher Clavius (1538–1612) is often associated with the astronomical and mathematical justification for shifting from the Julian to the Gregorian calendar. He was also a university professor who was convinced that mathematics should be a standard part of a university curriculum and who saw the need to train instructors of mathematics. He exerted his influence on the *Ratio Studiorum (The Plan of Studies)*, a 1599 document that included administrative norms and curricular guidelines for Jesuit schools as well as offering pedagogical suggestions in the form of "Rules" for teachers of various subjects (the Latin text **[15]** and English translations **[6, 7]** are available).

This 400 year old document provided guidelines for teaching mathematics in Jesuit schools in the seventeenth century and contains what may seem to be novel and even modern suggestions for student involvement in learning mathematics. In particular, the *Ratio Studiorum* called for what today might be called *student colloquia* and *interactive review sessions*. For this reason, Clavius might be considered the father of certain contemporary pedagogical techniques in mathematics.

The last forty years have seen numerous attempts to revitalize the mathematical curriculum and mathematical pedagogy, especially in the United States. Particularly in the last twenty years, much time and energy has been focused on university-level calculus courses, with such well-known results as the Harvard Reform Calculus approach (see also two volumes in the *MAA Notes* series **[5, 23]** and a book by Krantz **[13]** with its useful appendices). The Mathematical Association of America conducted special sessions on calculus reform at its annual meetings in 1996 and 1997. The October 1997 issue of the *American Mathematical Monthly* contained several articles on the topic and *The College Mathematics Journal* **[17, 10]** frequently addresses these concerns. Many websites, for instance at Cornell, Swarthmore, and Harvard, are devoted to this discussion.

The catch-phrase *cooperative learning* is often used to describe techniques such as student presentations and interactive classroom sessions. These interactive components arise from the experience, common among teachers, that one comes to a deeper understanding of a subject by explaining it to others. This experience has prompted instructors to assign essays on mathematical topics **[20]** or oral presentations in classes. Although such ideas may seem revolutionary, there are, in fact, references to these practices in the *Ratio Studiorum*. In particular, a guideline in the Rules for the Professor of Mathematics in the *Ratio* requires that students present a solution to "some famous mathematical problem" before an assembly of other students **[6**, p. 46; **7**, p. 175], a precursor of contemporary student presentations.

Christopher Clavius, Jesuit mathematician

Christopher Clavius was born in Bamberg, Germany in 1538. In 1555, he traveled to Rome to join the Society of Jesus (commonly known as the Jesuits), a religious order in the Catholic Church. Nine years later, in 1564, he was ordained a priest while finishing his theological studies at the Jesuit-run Roman College, now known as the Gregorian University. He began teaching mathematical subjects on a regular basis at the Roman College around 1564. Clavius taught for more than 45 years in Rome, until his death in 1612 (except for two years in 1596 and 1597 when he was in Naples and Spain). Documents from the Roman College [**24**, p. 7] indicate that Clavius was the sole teacher of mathematics for at least 22 years between 1564 and 1595.

Clavius lived at the beginning of what is now called the "Scientific Revolution." This was the time of Copernicus (who published *De revolutionibus* in 1543) and the time of Galileo (who was condemned by the Inquisition in 1633) with whom Clavius discussed astronomical phenomena. Newton would be born 30 years after the death of Clavius. These were times of great scientific speculation as well as intellectual inertia and ecclesiastical opposition, particularly to Galileo.

In the academic community of sixteenth century Italy, there were two factions in contention over the role of mathematics in science and education. One faction consisted of a number of prominent Italian philosophers who denied that pure mathematics should be regarded as *scientia*, that is, scientific knowledge in the Aristotelean sense [**3**, pp. 36–37; **12**, p. 5; **14**, p. 33; **24**, p. 136]. The other faction was represented by Clavius. At that time in Italy, the only accepted professional disciplines for university study were theology, medicine, and law [**8**, p. 126; **12**, p. 81]. Mathematical topics were presented in the study of logic in the *trivium* (alongside grammar and rhetoric) and also in the *quadrivium* of arithmetic, geometry, and astronomy (taught as applied geometry), and music (taught as applied arithmetic). But these were preparatory subjects to be learned by students before entering a university to study philosophy as a basis for professional studies. Mathematics itself was not usually taught in a university [**3**, p. 35]. Clavius attempted to change that, at least within Jesuit schools.

The Jesuit Order was approved in 1540 and almost immediately set about establishing schools. In 1556, the year of the death of Ignatius of Loyola, founder of the Jesuit Order, the number of Jesuit schools was 35. Fifty-nine years later, in 1615, three years after Clavius died, the number had grown to 372 [**11**, p. 201]. In the middle of this period, the creation of the *Ratio Studiorum* was authorized by the head of the Jesuit Order, Father General Claude Aquaviva, in order to provide common guidelines for the curriculum, pedagogy, and organization of these schools.

In 1584, six Jesuit teachers, elected from different European Jesuit provinces, gathered in Rome and reviewed the various local educational documents in use at Jesuit schools. In August 1585, this commission submitted a report to Aquaviva who, in turn, submitted the draft to the faculty of the Roman College, where Clavius had been teaching for some 20 years [**7**, pp. 28–30]. After revision, the first draft of *Ratio* appeared in 1586 and was sent to Jesuit schools throughout Europe with a request for comments. A second draft ("1586B") also was written in 1586 but was not published or circulated. The collected comments on the original 1586 *Ratio* were again studied by the faculty of the Roman College and a revised *Ratio* was published in 1591 to be circulated for review. After more study in Rome, both by Father General Aquaviva and the faculty of the Roman College, a definitive version was promulgated in 1599 [**7**, pp. 31–33] (also see the introductory letter promulgating the 1599 *Ratio*, [**6**, pp. xii–xiii; **7**, pp. 119–20]).

The *Ratio* provided a common list of subjects to be taught, including Latin, Greek, Hebrew, scripture, theology, philosophy (which included logic, physics, and meteorology), moral philosophy, and mathematics, forming an early type of core curriculum. In addition, it included various suggestions for pedagogical techniques and rules for teachers. By providing such guidelines, the *Ratio* sought to offer a uniform approach to secondary and higher education in schools associated with the Jesuit Order throughout Europe and elsewhere. The definitive version of the *Ratio* appeared in 1599 at a crucial time in the history of Jesuit education when there was a rapid increase in the number of Jesuit schools.

During the two decades before the 1599 *Ratio* was published, Clavius wrote several treatises on mathematics which were reviewed by Aquaviva and, most probably, by others involved in the drafting and revision of the *Ratio* [**12**, pp. 61, 64] (the Latin text of these documents is also available [**16**, pp. 109–22]). The two primary documents were *Modus quo disciplinæ mathematicæ in scholis Societatis possent promoveri (A Method of Promoting the Mathematical Disciplines in the Schools of the Society)* (written around 1582) and *De re mathematica instructio (On Teaching Mathematics)* (written prior to 1593) [**22**].

The first document (*A Method* ...) seems to have been written in response to a request by Aquaviva in 1582 that the faculty of the Roman College convey its feelings about the teaching of various subjects [**16**, p. 109]. In the suggestions Clavius makes, one sees a hint of the intellectual climate of the time. To change the common opinion regarding mathematics, Clavius recommends that "*invitandus erit magister [artium mathematicarum] ad actus solemniores, quibus doctores creantur et disputationes publicæ instituuntur* (the mathematics teacher should be invited to the more solemn events at which doctorates are conferred and public disputations held ...)" [**16**, p. 115]. (The implication is that in many places they were not so invited!) He then takes to task, in particular, instructors of philosophy since, as he writes, "*docent, scientias mathematicas non esse scientias* (they teach that mathematical sciences are not sciences)" [**16**, p. 116]. He says that teachers should encourage students to learn mathematics, impressing on them how important the discipline is [**16**, p. 117]. Clavius also argues that one cannot understand various natural phenomena without mathematics. It is evident that some of Clavius's advice was incorporated by the drafting commission into the various drafts of the *Ratio*. It is also possible to see some similarities between the praise given to mathematics in the first, 1586 draft of the *Ratio* and the praise given to mathematics in Clavius's first document.

The second document (*On Teaching Mathematics*) is cited in a 1593 decree on the topic of educating teachers of mathematics [**16**, p. 622], written by the rector of the Roman College, Robert Bellarmine, at the wish of Aquaviva. The document emphasizes how vital it is to train future mathematics teachers well. It also offers specific suggestions on what mathematical topics should be taught, namely Euclid, the spherical elements of Theodosius, the *Conics* of Apollonius, applied topics pertaining to astronomy and physics (by studying Archimedes), and algebra [**16**, p. 118]. These topics briefly summarize a more detailed list of mathematical topics Clavius prepared prior to 1581 (*Ordo servandus in addiscendis disciplinis mathematicis [The Order to be Observed in Teaching Mathematical Disciplines]*, [**16**, pp. 110–15]). This document also proposes how this training should be undertaken, by suggesting private study of mathematics for a full year after young Jesuit seminarians had completed their study of philosophy and before studying theology [**16**, p. 118]. Near its end, the document includes a cry of exasperation, "*cum egeamus magistris mathematices* (We need mathematics teachers)" [**16**, p. 118].

In both documents, we notice the tension between Clavius's promoting the importance of mathematics in a university curriculum and the position held by Italian philosophers, some of whom were also Jesuits. This tension is also evident in the various drafts of the *Ratio*. For example, the 1591 draft of the *Ratio* contains a curious section that mentions Clavius and then instructs administrators of schools to guard that the philosophy instructors not disparage the dignity of mathematics (a topic alluded to in *A Method* ...). It then ends with the statement, "*fit enim sæpe, ut qui minus ista novit, his magis detrahat* (often, the less one knows about such things, the more he detracts)" (1591 *Ratio Studiorum*, Rules for the Provincial Superior: On Mathematics, n. 44 [**15**, p. 236]).

Readers of the various drafts of the *Ratio* and the documents written by Clavius can see the influence of someone who values mathematics highly and who is concerned that the young Jesuit Order train students well in the mathematical sciences, an unpopular position in late sixteenth century Italy. The personal opinion of Clavius about mathematics seems best summarized in his words: "Since ... the mathematical disciplines in fact require, delight in, and honor truth ... there can be no doubt that they must be conceded the first place among all the other sciences" [**3**, p. 38]. This praise is similar to the well-known statement of Carl F. Gauss (1777–1855), that "mathematics is the queen of the sciences," but Clavius lived two centuries earlier!

Mathematics pedagogy in the *Ratio Studiorum*

It seems reasonable to conclude that Clavius had some effect on the *Ratio Studiorum*. Aquaviva and those who drafted and revised the *Ratio* not only would have read the treatises described above, but they would have solicited personal comments from faculty of the Roman College, such as Clavius. The fact that Clavius is mentioned in both the 1586 and the 1591 drafts is proof enough of his influence. As noted earlier, Clavius was for many years the sole professor of mathematics at the Roman College whose faculty reviewed drafts of the *Ratio*. What is common to the three drafts and the final text is an emphasis on mathematics that was unusual for that time in Italy. These texts prescribe that all students be taught mathematics for one year, specifically, Euclid and topics related to geography (that is, applied geometry), and the sphere. (Since Clavius had published several mathematical texts, further detail in the final version of the *Ratio* was not needed.) In fact, the earlier drafts of the *Ratio* also recommended a second year of mathematics for those interested in further study.

The earlier drafts of the *Ratio*, which specifically mention Clavius, also suggested certain practices to help students understand the subject better. As noted above, these practices may be best understood in terms of contemporary counterparts: the *colloquium* and the *review session*.

In the 1586B draft, the text pertaining to mathematics includes a recommendation for colloquia or presentations by students: "*Semel aut iterum in mense auditorum aliquis in magno philosophorum theologorumque conventu illustre aliquod problema mathematicum enarret, prius a magistro, sicut oportet, edoctus*. (Once or twice a month, let some one of the students explain in detail some famous mathematical problem before a large gathering of students of philosophy and theology, after first having been taught it thoroughly by the teacher, as necessary)" [**15**, p. 177]. In the final 1599 *Ratio*, this recommendation is rephrased: "*Singulis aut alternis saltem mensibus ab aliquo auditorum magno philosophorum theologorumque conventu illustre problema mathematicum enodandum curet; posteaque, si videbitur, argumentandum*. (Let [the professor of mathematics] arrange that every month or every other month some one of the students before a large gathering of students of philosophy and theology has some famous mathematical problem to work out and afterwards, if it seems well, to defend his solution)" [**15**, p. 402] (also [**6**, p. 46; **7**, p. 175]). One is tempted to see the shift from "once or twice a month" to "every month or every other month" as a concession to the voices of opposition raised by other faculty.

In the 1586B draft, there is also mention of an interactive review session in the following recommendation: "*In cuiusque etiam mensis sabbato uno, prælectionis loco, præcipua, quæ per eum mensem explicata fuerint, publice repetantur, non perpetua oratione, sed se mutuo percunctantibus auditoribus hoc fere modo: Repete illam propositionem. Quomodo demonstratur? Potestne aliter demonstrari? Quem usum habet in artibus et in reliqua vitæ communis praxi?* (On one Saturday of each month, in place of the lecture, let the principal points that during that month had been explained be publicly repeated, not in an uninterrupted speech, but with the students mutually asking questions of themselves, generally in this manner: 'Repeat that proposition.' 'How is it proven?' 'Can it be proven otherwise?' 'What use does it have in the arts or in the other practices of common life?')" [**15**, p. 177].

As a result of recommendations by a general meeting of Jesuit delegates in 1593–94 that the definitive *Ratio* be briefer [**7**, p. 32], the details about student interaction and the types of questions students should be asking were no longer included. But the definitive 1599 *Ratio* kept the "Repetition" (review session) as a standard practice in its recommendation: "Once a month and generally on Saturday in place of the lecture, let the principal points that have been explained during that month be publicly repeated." [**15**, p. 402] (also [**6**, p. 46; **7**, p. 175]).

Similar pedagogical practices such as a presentation by students (or "disputation") and reviews are also prescribed by the 1599 *Ratio* for other subjects (for instance, Philosophy and Theology [**6**, p. 20–21; **7**, p. 144–45], and Scripture [**6**, p. 32; **7**, p. 159]). What is especially notable is the inclusion of versions of these practices, modified for teaching mathematics, in the 1599 *Ratio*. Admittedly there is no direct evidence, either from the documents of Clavius on mathematics or from other documents, that Clavius originated these pedagogical practices. Nevertheless, the fact that his name appears in earlier drafts of the *Ratio* in conjunction with the listing of these practices (and of the more detailed questions proposed) and that, as a member of the faculty at the Roman College, he reviewed the drafts of the *Ratio*, suggests that he did exert his influence on the text in the definitive 1599 *Ratio*.

The guidelines found in the 1599 *Ratio*, as well as the sample student questions found in the earlier versions, encouraged regular student participation and recommended that as part of a review of the subject matter, students reflect on alternative proofs of mathematical theorems and the application of mathematics to other areas of life. Similar recommendations have reappeared in the last decade, but, it is interesting to note, at the core of some of these modern pedagogical suggestions is a tradition over 400 years old.

The legacy of Clavius

From a contemporary viewpoint, Clavius created relatively little original mathematics, but the time was not congenial for this to happen. Today we know that mathematicians need the right climate to nurture new developments. However, Clavius did prove geometrically that, given a regular polygon with $2n + 1$ sides, the base angle is n times the summit angle in an isosceles triangle formed by a vertex of the polygon and its opposite side. (Thus, for a pentagon, n equals 2, and a base angle would be twice the summit angle.) Two centuries later Gauss would repeat the proof algebraically and use the result to construct a regular 17-sided polygon by rule and compass [**18**, p. 25; **19**, p. 336].

Perhaps, the greatest legacy of Clavius was his influence on the definitive version of the *Ratio Studiorum* which led to the inclusion of mathematics as a standard subject taught in Jesuit schools [**3**, pp. 34–36]. This inclusion is especially significant, given the common opinion in Italian academic circles about mathematics at the time. This was particularly crucial because the founding of Jesuit schools essentially coincided with the beginning of the scientific revolution.

In *A Method* ..., Clavius even proposed establishing an "academy" for more advanced study in mathematics. Such a specialized academy was explicitly mentioned in the first two public draft versions of the *Ratio*, but in the definitive 1599 version of the *Ratio* this mandate was reduced to the suggestion that "private lessons" be given to those more inclined toward mathematics (Rules for Provincials, n. 20, [**6**, p. 8; **7**, p. 130]). Although a special advanced mathematics curriculum was not mentioned in the definitive *Ratio* for all schools, Clavius's vision of establishing a special school to train Jesuit mathematicians was nevertheless realized in Antwerp near the end of Clavius's life. In 1611 François d'Aguilon, S.J. founded a school for mathematics and was joined by Gregory St. Vincent, S.J. around 1616 who had studied under Clavius in Rome. During his life, St. Vincent researched mathematical concepts we would now see as precursors to the infinitesimal calculus. Clavius's vision of a special "academy" may be viewed as a precursor of honors, seminar, or directed study classes in contemporary schools where students with an exceptional talent and love for mathematics may have their mathematical curiosity nurtured and challenged.

One additional contribution of Clavius to the mathematical community of later generations was his set of mathematical texts. For example, in 1574 Clavius published *The Elements of Euclid*. This was not simply a translation, but a text containing Euclid's work as well as comments on it, many taken from previous commentators and editors, but also including Clavius's own criticisms and elucidations of Euclid's axioms. This text has been held up as a model and led to Clavius being called the *Euclid of the 16th century* [**4**, p. 311]. It was this text that the Jesuit missionary Matteo Ricci brought to China and translated into Chinese, giving the Orient its first exposure to Euclid. Ricci was a student of Clavius in the mid-1570s and, because of his scientific knowledge which, to a large extent, was imparted to him by Clavius, is one of the few Westerners still honored in China. Ricci's grave near Beijing is still a place of pilgrimage. Other major works of Clavius include *In Sphæram Ioannis de Sacro Bosco commentarius* (1581), *Epitome arithmeticæ practicæ* (1583), *Astrolabium* (1593), *Geometria practica* (1604), *Algebra* (1608), and *Triangula sphærica* (1611). These well-written texts were reprinted numerous times and widely used in Jesuit schools. They were also used by such mathematicians as Leibniz and Descartes (who studied at the Jesuit College of La Flèche) [**18**, pp. 7, 48; **3**, p. 34].

In his texts, Clavius used notation that has become standard in mathematics today, such as the square root sign, parentheses, the x-like symbol for an unknown [**2**, pp. 369, 151, 154,], and the decimal point [**9**; **1**, p. 303], although Cajori seems less certain about Clavius's understanding of the period as a decimal point [**2**, p. 322]. Although some of these symbols had been used by others in German lands earlier, Clavius helped to bring these symbols to Italy and to standardize this mathematical notation due to their appearance in his widely-disseminated books.

Summary

The historian of science George Sarton calls Clavius, "the most influential teacher of the Renaissance" [**21**, p. 70]. Clavius was convinced of the importance of the study of mathematics in universities and helped to influence the Jesuit *Ratio Studioum* to include mathematics as a required part of the curriculum. This 400-year old document also advanced the insight that students would understand mathematical concepts better by explaining them to others and by questioning mathematical derivations during a review. Such practices are still effective tools for helping students of the twenty-first century appreciate the beauty and wonders of mathematics.

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Counting Trash in Poker

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On a regular basis, faculty members from the University of North Dakota (UND) get together to relax and socialize, that is, play poker. This paper grew out of an occurrence at one of our games.

First, we cover the necessary background material and then describe how this work came about. Next we discuss poker with hands of two or three cards and show that they are not natural for the standard 52-card poker deck or any possible deck variation. Finally we investigate poker hands with 4 or more cards and determine all deck parameters that lead to interesting games. You might use one of these variations to add spice to your regular poker night.

Background We have at UND a sophomore-level discrete math course, primarily designed to prepare computer science majors for a course in data structures. Since I am a combinatorialist, I teach the course often. Usually as extra-credit homework, or as an in-class group project, my students complete the following assignment.

> A standard 52-card poker deck has cards in 13 ranks Ace (A), 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack (J), Queen(Q), and King(K). There are four cards in each rank, one from each suit—spades(♠), hearts(♡), diamonds(◇), and clubs(♣). The ranks are ordered almost linearly A, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A—A being both high and low. The object is to form the best 5-card hand. The ranking of hands from high to low appears below. A description of each type of hand is included.
>
> Straight Flush: Five cards from the same suit in five consecutive ranks. Examples include A, 2, 3, 4, and 5 of spades and 10, Jack, Queen, King, Ace of diamonds, but not Q,K,A,2, and 3 of hearts.
>
> Four-of-a-Kind: Four cards from a single rank plus some other card.
>
> Full House: Three cards from one rank and two cards from another rank.
>
> Flush: Five cards from the same suit, but not a straight flush.
>
> Straight: Five cards in five consecutive ranks, but not all of the same suit.
>
> Three-of-a-Kind: Three cards from one rank and two cards each from different and distinct ranks, that is, not Four-of-a-Kind, and not a Full House.
>
> Two Pair: Two cards from each of two distinct ranks and another card from a third rank.
>
> One Pair: Two cards from some rank and three cards each from distinct and different ranks.
>
> Trash: Any hand not described above.
>
> **Assignment** Show that the ranking of hands is in increasing order of frequency.

The students are to use basic counting techniques to arrive at a solution equivalent to the one given in TABLE 1.

TABLE 1: 5-card poker hands from a standard 52-card deck

Hand	Frequency
Straight Flush	$\binom{10}{1}\binom{4}{1} = 40$
4-of-a-Kind	$\binom{13}{1}\binom{4}{4}\binom{48}{1} = 624$
Full House	$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{2} = 3{,}744$
Flush	$\binom{4}{1}\binom{13}{5} - 40 = 5{,}108$
Straight	$\binom{10}{1}\binom{4}{1}^5 - 40 = 10{,}200$
3-of-a-Kind	$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 = 54{,}912$
Two Pair	$\binom{13}{2}\binom{4}{2}^2\binom{11}{1}\binom{4}{1} = 123{,}552$
One Pair	$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3 = 1{,}098{,}240$
Trash	2,598,960 − sum of above = 1,302,540
Total	$\binom{52}{5} = 2{,}598{,}960$

We wish to point out that the way that most people first count trash is exactly as described in the penultimate row of the table. Also, we clarify that in addition to the given hierarchy there is a sub-hierarchy of the hands based on the almost linear ordering of the ranks. According to this sub-hierarchy, a K-high straight beats a Q-high straight, etc. Consequently we've counted royal flushes (A-high straight flushes) as straight flushes since they are the highest such hand based on the sub-hierarchy. This sub-hierarchy is exactly what is used to determine which player has the highest hand, that is, who is the winner.

One point of the exercise is that the hierarchy of hands based on frequency is logical. In this hierarchy, hands with value have structure—the more structure, the greater the value. Here structure only takes the form of multiple holdings of some rank(s), or all cards in consecutive ranks, or all cards from the same suit, or all cards consecutive and from the same suit. Some might argue to include 4-straights, 4-flushes, etc., as hands with structure. This is not our convention, nor the norm in most poker games south of Canada.

A second point is that the frequencies are relatively well separated. To clarify, we define the *relative separation* of two consecutive hands in the hierarchy to be the absolute value of the difference of their frequencies divided by the average of their frequencies. So, for example, the relative separation of trash and one pair is $204,300/1,200,390 \simeq 0.17$, whereas the relative separation of flush and full house is $1,364/4,426 \simeq 0.31$. Good relative separation is important since it averts arguments over which types of hands should have higher value (and therefore may save lives).

At some point, someone noticed that a revision in the hierarchy can be also used to determine rigorously which player has, in some sense, the lowest hand. So, for example, one pair is lower than two pair. Some players take a 5-high straight to be the lowest hand. Our convention is that a low hand should have as little structure as possible. Therefore a trash hand is lower than any straight, flush or straight flush, and also lower than any hand containing a pair. Next, ties among trash hands as low hands can be broken using the sub-hierarchy in reverse order with A always low. So K, J, 10, 9, A beats K, Q, 4, 3, 2. TABLE 2 displays the partitioning of trash hands into low-ball hands.

TABLE 2: 5-card trash hands from a 52-card deck as low-ball hands

Hand	Frequency
K-Low	502,860
Q-Low	335,580
J-Low	213,180
10-Low	127,500
9-Low	70,380
8-Low	34,680
7-Low	14,280
6-Low	4,080

To count the number of 9-low hands, we know that the hand contains a 9 as the highest rank. The remaining four cards must be from four distinct ranks below 9, but not 8,7,6, and 5. For each rank in the hand we select a suit, we just can't pick the same suit for each rank. So by the multiplication principle there are $\left[\binom{8}{4} - 1\right]\left[4^5 - 4\right] =$ 70,380 9-low hands. Similarly there are $\left[\binom{6}{4} - 1\right]\left[4^5 - 4\right]$ 7-low hands. The only hitch occurs at K-low where there are $\left[\binom{12}{4} - 2\right]\left[4^5 - 4\right]$ hands. The adjustment is made because there are two ways to make a straight with a K and four lower ranks: 9, 10, J, Q, K and A, 10, J, Q, K. Summing the frequencies in the table gives rise to a second method for counting trash hands.

The previous paragraph also leads us to a third way to count trash: build a hand with no pairs, subtract all flushes, subtract all straights, and by inclusion-exclusion add straight flushes back in. So the number of 5-card trash hands from a standard 52-card deck is

$$\binom{13}{5}4^5 - \binom{13}{5}4 - \binom{10}{1}4^5 + \binom{10}{1}4 = \left[\binom{13}{5} - 10\right][4^5 - 4].$$

I am indebted to the students from group 3 for project number 5 in M208, section 2, fall 1999, for this approach.

Whereas a professional poker player would prefer not to split the pot, in a social game, in order to keep more players "in a game," the cheapest trick is to have the high hand split the pot (as evenly as possible) with the low hand. Since there is no clear advantage to trying to win high or low, about the same number of players should try to win each direction. Thus the split should be even.

Finally, we remark that the frequency of trash hands is almost exactly half of the total. It's not surprising therefore that, in practice, a trash hand almost always wins the low, and a nontrash hand almost always wins high. So with our conventions of structure, and nonstructure, it is quite natural to equate trash hands with low hands.

If the hands with structure were extended to include A-high, etc., or if a nontrash hand had better than 50% chance of winning the low-half of the pot, it would not be natural to equate trash hands with low hands. We think this would make it less likely that high-low, split-pot games would be as ubiquitous as they are in social poker games.

Self-destruction

This work originated when one of the players, who shall remain anonymous, tried to mess with perfection. The name of one of the games we commonly play is not suitable for print, so we'll call it Self-destruction. It was invented in 1982 by Professor Andrew Woldar of Villanova University when he was a graduate student at Ohio State University [**4**].

Each player is to make two 5-card hands (usually one high and one low, though they may be both high or both low) given ten cards. Cards are dealt two at a time face up in five dealing rounds. Rounds proceed clockwise from the dealer who begins by giving two cards to the first remaining player. That player must decide whether the two cards go into the same hand or different hands. Once placed in a hand a card may not be moved to the other hand later. After a placement decision is made, the dealer continues to the next remaining player. After every dealing round, except the last one, there is a round of betting based on the earliest example of the highest hand held. A player may choose to fold by not participating in the betting and not continuing to the next round. After the last dealing round the player with the highest hand splits the pot with the player who holds the lowest hand (sometimes the same player takes all). Since neither hand may contain more than five cards, some hands are forced. This explains the name: If the two hands don't fit together in the most advantageous way, you have only yourself to blame.

The only drawback to the game is that the total number of players must be fewer than six, or there might not be enough cards to go around.

One night one of our crew (the electrical engineer) wanted to play Self-destruction, but we had seven players. His solution was to call a game based on hands of size three, instead of size five. This would require at most $7 \cdot 2 \cdot 3 = 42$ cards. Instantly my intuition told me that this was a bad idea. Of course I'll try almost anything once. So we played. Afterward the feeling that this was not a good idea lingered, so I set out to figure out why—in hopes of explaining to my friend the error of his ways.

This was not a new situation. We had previously had players who called wild card games. With most I quietly shared the article "Poker with wild cards, a paradox?" by Steve Gadbois [**2**]. Reading it helped put them back on the path of righteous poker playing—no wild card games. Now all I needed to do was to convince my friend that playing poker with three-card hands is not natural.

Poker games with 2-card and 3-card hands

The purpose of this section is to argue that poker played with hands of size three is unnatural, and that poker should not be played with hands of size two. Of course I wanted to prove that poker should not be played with hands consisting of three cards.

I thought that I might be finished when I read the title of the first paper referenced by Gadbois [**2**], namely, "Why poker is played with five cards" [**1**]. Unfortunately Cheung [**1**] only showed why poker using a standard 52-card deck should not be played with 4-card hands, 6-card hands or 7-card hands.

I was quickly able to generate TABLE 3, which can be used to show that Self-destruction with 3-card hands is not natural. The first thing to notice is that the relative

TABLE 3: 3-card hands from a standard 52-card deck

Hand	Frequency
3-of-a-suit-in-a-row	48
3-of-a-kind	52
3-in-a-row	720
3-of-a-suit	1,096
One pair	3,744
Trash	16,440
Total	22,100

separation of the top two types of hands is not great (.08). So, there may well be arguments as to whether a high three-of-a-kind should beat a low straight flush. Second, we see that an individual is about three times as likely to get a trash hand as a nontrash hand. In this scenario we cannot naturally equate trash hands with low hands. Thus high-low, split-pot poker games are not as natural in this scenario.

So we're done. Right?

Well, not exactly. What we really know is that if we play poker with a standard 52-card deck, then hands of size three would not lead as naturally to high-low, split-pot games. We still don't know that this is true for any size deck! After all the poker deck originated from the tarot deck. Who's to say there isn't a better deck design? We could have more or fewer suits, and more or fewer ranks. There seems to be an endless supply of possibilities. How do we analyze them all? What we need is a way to single out only the most promising candidates. This means we need to know the definition of a promising candidate.

First of all, any game should have the property that hands counted as high should be those with structure, more structure implying more value. So our high hands should include such structures as pairs, triples, or quadruples of the same rank, and combinations of these. Also, there should be the analog of straights, flushes, and straight flushes.

For any tie-breaking sub-hierarchy to have a chance there should be the same number of cards of each rank, and the same number of ranks for each suit. So let n be the number of ranks, r the number of suits, and k the number of cards in a hand. Also let us suppose that there is a rank A, which is both high and low, and that the remaining ranks are ordered linearly in between. So to summarize, nontrash hands have at least one pair of the same rank, or are k-of-a-suit-in-a-row (straight flush), or k-in-a-row (straight), or k-of-a-suit (flush).

Next we need to consider the size of the deck. If the total number of cards, nr, in relation to hand size k is too small we'll be restricted as far as how many people can play. So let us suppose that $nr/k \geq 7$. Also we don't want the deck to be so large that it cannot be easily shuffled. Realistically most players can manipulate four standard decks; this seems to be large enough. So let us also suppose that $nr \leq 208 = 4 \cdot 52$. (Realize that these constraints on deck size are only one possible choice.)

Finally, for high-low, split-pot games to occur naturally, we would like the game to be balanced nontrash to trash. That is, we would like the total number of trash hands to be about half of the total. This means that there should be some trash hands, so we must have more than one suit (or else every hand is a flush). We must also have $1 < k < n$, since if $k = 1$ every hand is one-of-a-rank-of-a-suit-in-a-row, and if $k \geq n$ then every hand either contains a straight or at least one pair by the pigeon-hole principle.

With these conventions and constraints, to count k-card trash hands we first count all hands with no pairs, subtract all straights, subtract all flushes and add back in straight flushes. So if f is the number of trash hands, $f = (r^k - r)\left[\binom{n}{k} - (n+2-k)\right]$. Moreover we want f to be about half the size of $\binom{nr}{k}$, which is to say that we want the probability of a hand being trash to be near 0.5. After simplification we find that the probability of trash is

$$p(n, r, k) = \frac{(r^k - r)[\mathrm{P}(n, k) - k!(n+2-k)]}{\mathrm{P}(nr, k)},$$

where $\mathrm{P}(m, l) = m!/(m-l)!$, the number of ways to arrange l elements from a set with m elements.

Realize that for the formula for p to make sense, only k needs to be integral. So for a fixed k the graph of $p(n, r, k)$ is a surface in $\mathbb{R}^3$. Since $2 \leq r, n$ and $p \geq 0$ we are interested only in that part of the surface in the first octant. If we wanted to know where p was exactly equal to 0.5, we would intersect our surface with the plane $p = 0.5$ and look at the n, r-coordinates of the shadow of the resulting curve in the n, r-plane to see if any were integral.

Thus to get close to $p = 0.5$, we want to slice our surface with two planes parallel to the n, r-plane separated by a distance of $2 \cdot \epsilon$ and centered about $p = 0.5$. And then ask when the shadow in the n, r-plane of the resultant pipe-cleaner shape contains points with integral coordinates. Clearly the choice for ϵ should be large enough so that the classical parameter set (13, 5, 4) survives (with $p(13, 4, 5) = 0.501177$), yet small enough to allow us to pare down the total number of parameter sets we want to investigate.

One good choice for ϵ might be 0.005, since, in a high-low split-pot game after rounding to the nearest cent, the high hand and low hand would each expect 50 cents for each dollar in the pot. In case the pot did not split evenly the hand whose type has lower probability would take the extra amount.

Concentrating on the 3-card hand first, we have

$$p(n, r, 3) = \frac{(r^2-1)(n-1)(n^2-2n-6)}{n(nr-1)(nr-2)}.$$

By plotting the curves $p(n_0, r, 3)$ for fixed integers $n_0 \geq 4$, we find that investigating the curve $p(7, r, 3)$ will be sufficient. Now we find that $p(7, 2, 3) = 0.478022$, $p(7, 3, 3) = 0.523308$, $p(7, 4, 3) = 0.531136$, and $p(7, 5, 3) = 0.531704$. Then as r approaches ∞, $p(7, r, 3)$ approaches 174/343 from above. To six decimal places $174/343 = 0.507289$. So the points on $p(n, r, 3)$ with integral inputs $n \geq 4$ and $r \geq 2$ are never within 0.005 units of 0.5.

Even if we double the size of ϵ to 0.01 we find that the smallest positive integral value of r for which $|p(7, r, 3) - 0.5| < \epsilon$ is 79. So we would need a deck with 553 cards. (EEEeeek!) Thus it would seem that the development of social, high-low, evenly split-pot games would not have been as natural if hands contained three cards.

The 2-card hand is radically different. To begin with, the formula for $p(n, r, 2)$ reduces to $p(n, r, 2) = (r-1)(n-3)/nr - 1$. Notice that $p(n, r, 2) = 1/2$ is a hyperbola that passes through three lattice points. Two of these solutions, (11, 3, 2) and (7, 7, 2), are within our realm of interest. In addition one can show that $p(n, 2, 2)$ asymptotically approaches 1/2 from above and $p(6, r, 2)$ asymptotically approaches 1/2 from below. Whereas $p(n, r, 3)$ stays clear of the value 1/2 for positive integral inputs, $p(n, r, 2)$ gets arbitrarily close infinitely often for positive integral inputs.

However perfect balance does not imply perfect separation. For the game with parameters (11, 3, 2) the number of straight flushes is equal to the number of one pair hands (33).

Sometimes it appears that perfect balance and perfect separation coincide. For the game with parameters (7, 7, 2) the probabilities of getting a straight flush, flush, one pair, straight, and trash are respectively 1/24, 1/12, 1/8, 1/4, and 1/2. It is hard to imagine a better scenario. But before we call Hoyle to tell them they need to start building 49-card decks, we should make sure that it's safe to play with a few cards shy of a full deck. Unfortunately we find that for this game the frequency of 6-low hands equals the frequency of 7-low hands (168). This collision is unavoidable and only requires $k = 2$.

THEOREM 1. *In a deck with n ranks ordered* $A, 2, 3, \ldots, Y, X, A$, *the number of 2-card* Y*-low hands is the same as the number of 2-card* X*-low hands.*

Proof. The number of 2-card Y-low hands is

$$\left[\binom{n-1-1}{1}-1\right][r^2-r] = (n-3)(r^2-r).$$

Since X is playing the role of K, the number of 2-card X-low hands is

$$\left[\binom{n-1}{2}-2\right][r^2-r] = (n-3)(r^2-r). \qquad \blacksquare$$

So there are 2-card poker games with perfect balance and perfect separation with respect to a standard hierarchy based on frequency, but there is always an unavoidable collision in the sub-hierarchy. This is why poker shouldn't be played with 2-card hands.

Vanilla, crunch, separation anxiety, and survivors

It begins to appear that the standard game of poker is pretty special. But just to be sure, let's go ahead and look at hands with 4 or more cards. To prevent us from missing any interesting games let us also take $\epsilon = 0.01$. The following *Mathematica* commands were used to generate all parameter sets that satisfy these constraints and our constraints on the size of the deck.

```
Prm[m_ ,k_ ]:=Product[i, {i, m − k + 1, m}];
Pb[n_,r_,k_]:=N[(r^k − r)(Prm[n, k] - Factorial[k](n + 2 − k))/(Prm[nr, k])];
m := 0;
Do[  If[Abs[Pb[n, r, k] − 0.5] < 0.01,
     If[n ∗ r/k > 7,
       m = m + 1;
       Print[{n, r, m,Pb[n, r, k]}]]],  {k, 4, 18}, {n, k + 1, 104}, {r, 2,Floor[208/n]}]
```

We find that a total of 78 parameter sets satisfy our constraints. Notice that the constraint $nr \leq 208$ forces a decline in number of possibilities as k increases. Of the 78 parameter sets we observe that about two-thirds have $r = 2, 3, 4$, or 5. The remaining parameter sets have $k = 4, 5, 6$, or 7. We tried to produce relevant computations along these two lines.

When r is fixed and relatively small we have a lot of knowledge of the types of hands. For example, when $r = 4$ the hands with structure are "i four-of-a-kind with j three-of-a-kind and l pair," where $0 \leq i \leq \lfloor k/4 \rfloor$, $0 \leq j \leq \lfloor (k-4i)/3 \rfloor$, $0 \leq l \leq \lfloor (k-4i-3j)/2 \rfloor$, and $i + j + l \neq 0$ (that is, not all zero); "straight;" "flush;" and "straight flush." The frequencies of these hands are functions of n and k. When these functions are defined in *Mathematica*, we can generate lists of frequencies for specific

n and k. For our example of $r = 4$, we have the following examples of *Mathematica* functions:

```
ftr[n_ ,k_ ]:=(Binomial[n,k] - (n+2-k)) * (4^k - 4);

f432[n_ ,k_ ,i_ ,j_ ,l_ ]:=Binomial[n,i]*Binomial [n-i,j]*4^j* Binomial
[n-i-j,l]*6^l* Binomial [n-i-j-l,k-4*i-3*j-2*l] * 4^(k-4*i-3*j-2*l);

fsf[n_ ,k_ ]:=(n+2-k) * 4;

fst[n_ ,k_ ]:=(n+2-k) * (4^k - 4);

ffl[n_ ,k_ ]:=(Binomial[n,k] - (n+2-k)) * 4;
```

For a given value k_0 of k we often found that the parameter set (n, r, k_0) satisfied our constraints for certain integral intervals of n, $n_{min} \leq n \leq n_{max}$. So we were able to use a loop in *Mathematica* to generate lists of hands and their frequencies for a fixed k_0.

Since our method for counting trash does not use the total number of hands, we can check each computation by comparing the sum of frequencies to the quantity `Binomial[4*n, k0]`.

We subsequently investigated each case for separation. We define the *separation index* of a parameter set to be the minimal relative separation of consecutive types of hands after sorting the list of frequencies into decreasing order. The following *Mathematica* commands carry out this computation, where y is the list of frequencies.

```
x=Sort[y,Greater]
w=TABLE[rs[x[[i]],x[[i+1]]],{i,1,Length[x]-1}]
si=Min[w]
```

Here `rs[a_ ,b_ ]:=N[2*Abs[a-b]/(a+b)]` is the *Mathematica* function that computes the relative separation of two frequencies.

The other line of approach has k fixed and relatively small. Now it is possible to generate an exhaustive list of types of hands. For each type of hand its frequency is a function of n and r. So, for example, when $k = 6$ the number of 2-pair hands from a deck with n ranks and r suits is $\binom{n}{2}\binom{r}{2}^2\binom{n-1}{2}r^2$. Of course, not every possible hand occurs. For example, when $k = 5$, 5-of-a-kind is a possible hand, but its frequency is zero when $r < 5$. More formulas are given by Cheung **[1]**

As before, these functions of n and r can be defined in *Mathematica* and used to generate lists of hands with their frequencies. Lists of frequencies were investigated as above for separation.

Our results fell roughly into two categories: parameter sets representing good games (survivors), and parameter sets representing not-so-good games. The not-so-good games fell into three sub-categories: dull and unexciting (vanilla), no separation between two types of hands (crunch), and very little separation between two types of hands (separation anxiety).

Let G be a poker game with parameters (n, r, k) that satisfy our constraints. Let us call G a *vanilla game* if $r = 2$. Let us define a *crunch game* as one with separation index equal to 0. Finally, we say G has *separation anxiety* if its separation index is less than or equal to 0.08 (a little less than half of the separation index of the standard game).

The reason that every game G with parameters $(n, 2, k)$ is vanilla is that if $j = \lfloor k/2 \rfloor$, then the possible hands are trash, 1 pair, 2 pair, ..., j pair, flush, straight, and straight flush. One can show that, for each game under consideration with parameters $(n, 2, k)$, the hierarchy of hands based on frequency is 1 pair, 2 pair, ..., i pair, flush,

$i+1$ pair, ..., j pair, straight, straight flush where $i = \lfloor k/4 \rfloor + 1$. (We did not seek to prove this in general.)

Moreover, except when $k = 15$, the separation index of each game with parameters $(n, 2, k)$ is at least 0.2 and occurs as the relative separation of trash and 1 pair. It's only mildly interesting that the game with parameters $(81, 2, 15)$ has separation anxiety (0.072573) and that the separation indices of the games with parameters $(82, 2, 15)$, $(83, 2, 15)$, and $(84, 2, 15)$ occur at flush and 5 pair.

Our last argument along these lines is that the increases in deck sizes for these games don't pay off in terms of number and variety of types of hands. When $k = 10$ or 11 the games have the same number of types of hands as regular poker, nine, with much less variety. At $k = 12$ we've approximately doubled the size of the deck, but have only ten basic types of hands. Altogether there are 25 vanilla games among the 78 generated.

Next we come to those games that are crunch games. The following parameter sets generated games with one or more occurrence of two distinct hands having the same frequency: $(9, 5, 4)$, $(20, 5, 6)$, $(23, 3, 7)$, $(27, 5, 7)$, $(35, 5, 8)$, $(36, 5, 8)$, $(38, 3, 9)$, $(39, 3, 9)$, $(57, 3, 11)$, $(58, 3, 11)$, $(59, 3, 11)$, $(67, 3, 12)$, $(68, 3, 12)$ and $(69, 3, 12)$. We noticed that it often is the case that a collision is independent of the parameter n. In fact we can prove several theorems of the following sort.

THEOREM 2. *Let G be a poker game with parameters $(n, 5, k)$ and let l be a nonnegative integer. The frequency of the hand 3-of-a-kind and l pair is nonzero and the same as the frequency of the hand $l+2$ pair if and only if $k = (l^2 + 7l + 8)/2$.*

Proof. Let g be the function of n and k that gives the frequency of the hand 3-of-a-kind and l pair using 5 suits. Also, let h denote the function of n and k that gives the frequency of the hand $l+2$ pair using 5 suits. We have

$$g(n,k) = \binom{n}{1}\binom{5}{3}\binom{n-1}{l}\binom{5}{2}^{l}\binom{n-l-1}{k-3-2l}\binom{5}{1}^{k-3-2l},$$

which simplifies to

$$g(n,k) = \frac{10^{l+1}5^{k}}{5^{3+2l}}\frac{P(n, k-2-l)}{l!(k-3-2l)!}.$$

Whereas

$$h(n,k) = \binom{n}{l+2}\binom{5}{2}^{l+2}\binom{n-l-2}{k-2l-4}\binom{5}{1}^{5-2l-4},$$

which simplifies to

$$h(n,k) = \frac{10^{l+2}5^{k}}{5^{4+2l}}\frac{P(n, k-2-l)}{(l+2)!(k-4-2l)!}.$$

So after massive amounts of cancellation (which we may do if both are nonzero), we have that $g(n, k) = h(n, k)$ if and only if $2(k-3-2l) = (l+2)(l+1)$ if and only if $k = (l^2 + 7l + 8)/2$. ∎

For each nonnegative integer l the expression $l^2 + 7l + 8$ is positive and even, so every k is integral.

We draw the immediate corollary that there are infinitely many crunch games with 5 suits. Worse news still is the following theorem.

THEOREM 3. *Let G be a poker game with parameters $(n, 5, k)$ and l be a non-negative integer. The frequency of the hand 4-of-a-kind plus l pair is nonzero and the same as the frequency of the hand two 3-of-a-kinds plus $l-1$ pair if and only if $l = 1$, independent of n and k.*

Proof. Let g be the function of n and k that gives the frequency of 4-of-a-kind plus l pair using 5 suits. Let h be the function of n and k that gives the frequency of two 3-of-a-kinds and $(l-1)$ pair using 5 suits. Then

$$g(n,k) = \binom{n}{1}\binom{5}{4}\binom{n-1}{l}\binom{5}{2}^{l}\binom{n-1-l}{k-4-2l}\binom{5}{1}^{k-4-2l},$$

which simplifies to

$$g(n,k) = \frac{10^l 5^k}{5^{4+2l}} \frac{P(n, k-l-3)}{(l!)(k-4-2l)!}.$$

Whereas

$$h(n,k) = \binom{n}{2}\binom{5}{3}^2\binom{n-2}{l-1}\binom{5}{2}^{l-1}\binom{n-2-(l-1)}{k-6-2(l-1)}\binom{5}{1}^{k-6-2(l-1)},$$

which simplifies to

$$h(n,k) = \frac{10^{l+1}5^k}{5^{5+2l}} \frac{P(n, k-l-3)}{2(l-1)!(k-4-2l)!}.$$

If we set g and h equal and cancel terms we find $l! = (l-1)!$. This means $l = 1$. ∎

When $l = 0$ in Theorem 2, we have $k = 4$, so for $n > 4$ every game with parameters $(n, 5, 4)$ is a crunch game. When $n > k \geq 6$, Theorem 3 tells us that any game with parameters $(n, 5, k)$ is a crunch game. We know we don't want $k = 2$ or 3, so if $r = 5$ we must have $k = 5$. No such game survives for our choice of ϵ; therefore we should not play poker with 5 suits!

The theorems derived from the collisions with $r = 3$ have a slightly different flavor. We display them without proof.

THEOREM 4. *Let G be a poker game with parameters $(n, 3, k)$ and l be a positive integer not divisible by 3. The frequency of the hand 3-of-a-kind and l pair is nonzero and equal to the frequency of the hand $l+2$ pair if and only if $k = (l^2 + 9l + 11)/3$.*

THEOREM 5. *Let G be a poker game with parameters $(n, 3, k)$ and l be a positive integer not divisible by 3. The frequency of the hand 3-of-a-kind with $l+2$ pair is nonzero and equal to the frequency of the hand two 3-of-a-kinds and l pair if and only if $k = (l^2 + 15l + 38)/6$.*

Of the 78 parameter sets of the games that satisfy our constraints those with parameters (15, 10–13, 5), (17, 3, 6), (21, 9, 6), (25, 4, 7), (29, 7, 7), (33–34, 4, 8), and (47–48, 3, 10) yield games with separation anxiety. Technically one could play a game with separation anxiety, but the lack of separation may lead to arguments as far as the hierarchy of hands.

All that remains are the 27 survivors. The parameter sets of these games are (9, 4, 4), (10, 12–20, 4), (12, 3, 5), **(13, 4, 5)**, (14, 6–7, 5), (15, 9, 5), (19, 4, 6), (20, 6, 6), (21, 7–8, 6), (26, 4, 7), (28, 6–7, 7), (30–31, 3, 8), (42–43, 4, 9), and (52, 4, 10).

In addition to the standard game, there are a number of these games we can play by systematically building an appropriately sized deck using one or more standard decks.

For example, to play a game with parameters $(n, r, k) = (9, 4, 4)$, which we might call "French Revolution," we take a normal deck, eliminate the royalty (Js, Qs, Ks, and tens since we're not sure if they're royalty) and play with 4-card hands.

Or we might choose to play "Blood and Guts" with parameters $(12, 3, 5)$. Now we take a normal deck, kill the kings and cut out all the hearts.

Another fine example is "Quadruple Melting Pot." We start with four normal decks with the same back design and color. We label suits $\spadesuit_1, \spadesuit_2, \spadesuit_3, \spadesuit_4, \heartsuit_1, \heartsuit_2, \heartsuit_3, \heartsuit_4$, etc. Then since a melting pot has no royalty we eliminate the Jacks, Queens and Kings. So we have a deck with parameters $(10, 16, 4)$. Notice that we can also play "Triple Melting Pot" and "Quintuple Melting Pot."

If you prefer, you can play "Bloody Triple Melting Pot." This is Triple Melting Pot with the $\heartsuit$s ripped out. Instead of labeling suits differently, we keep suit values but change ranks to $A_1, 2_1, 3_1, \ldots 10_1, A_2, 2_2, \ldots, 10_2, A_3, \ldots, 10_3, A_1$. So we get a game with parameters $(30, 3, 8)$.

As it turns out, standard poker is not necessarily the optimal 5-card poker game. A game with better balance and separation is "Double Bloody Super-Ace." We start with two normal decks, label the suits $\spadesuit_1$, $\spadesuit_2$, etc., cut out all the $\heartsuit$s and add a Super-Ace, AA, to each remaining suit so our ranks are AA, A, 2, 3, ..., K, AA. The parameters are $(14, 6, 5)$.

It seems strange that no entry has $r = 5$, yet except for $k = 8$, we can play a game with $r = 4$ for $4 \leq k \leq 10$ (sometimes more than one). We've already seen the standard game, and French Revolution.

When $k = 6$, we would like 19 ranks to go with our 4 suits. There was not an obvious joke to go with this situation, so we thought of calling this "Serious Poker." Then, since this must be a purely academic game, the answer occurred to R.P. Millspaugh. To play "Academic Poker," label the ranks Professor, Kindergartner, First-grader,..., Twelfth-grader, Frosh, Sophomore, Junior, Senior, Graduate Student, Professor. Notice that Professor is both high and low. Can we therefore deduce that our students will never ace anything?

When $k = 9$ we might call the game "Triple Super-Ace." We start with three normal decks. To each suit of each deck we add a super-ace, AA. We order the ranks $AA_1, A_1, 2_1, \ldots, K_1, AA_2, A_2, \ldots, K_2, AA_3, A_3, 2_3, \ldots, K_3, AA_1$. So we obtain a game with parameters $(42, 4, 9)$.

There does seem to be a pattern for doubling. If we double the number of ranks we increase the size of the hand by about one-and-a-half times. So we get "Double Poker" with parameters $(26, 4, 7)$ by using two normal decks and ranks $A_1, 2_1, \ldots, K_1, A_2, 2_2, \ldots, K_2, A_1$.

Also we have "Double (Double), Toil and Trouble" with parameters $(52, 4, 10)$. There are 26 types of hands in this game so the title of the game also describes memorizing the hands and their place in the hierarchy based on frequency.

This example would intuitively reinforce our restricting the size of the deck. As the total number of cards increases, so too may the number of types of hands. We can clearly produce games that are not reasonably playable by humans. They would therefore hold less interest. Still, it is nice to know that poker is not a unique game. There are alternatives, if we wish to play them. Also there are several questions that remain to be answered.

In the standard game we can argue along the lines set out in [**2**] that no wild-card game makes sense. But is this true in general?

It seems as though there are too many possibilities to explore.

Acknowledgment. The author wishes to express appreciation to the referees for many valuable suggestions. Thanks also go to the guys in my poker group, without them this could not have happened. Most importantly the author thanks his wife for allowing him to play—poker that is.

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4 or 4? Mathematics or Accident?

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Readers may be unaware of the relationship between divisibility properties of numbers and topological properties of the numerals used to represent them. Let us use the word *number* for both the concept and the numeral. Given a number n, a proper prime divisor of n is a prime number, different from n, that divides n evenly. Note the following:

THEOREM. *Consider the set of numbers* $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. *For* $a, b \in \mathcal{S}$ *the following are equivalent:*

(i) a and b have the same number of proper prime divisors (counted with multiplicities)

(ii) a and b have the same homotopy type.

(iii) a and b cut a sheet in the same number of pieces if you write them down with a scalpel.

COROLLARY. *A number in* $\mathcal{S}$ *(other than* 1*) is prime if and only if it has trivial homotopy type or, equivalently, it does not tear the sheet in separate pieces.*

We now answer the first question in our title:

COROLLARY. *The correct symbol for the number four is* 4, *not* **4**.

Although the equivalence between *(ii)* and *(iii)* in the theorem depends on Alexander-Pontrjagin duality, this note, in spirit, belongs to the so called *Shape Theory* introduced by K. Borsuk. Readers may be interested to read Borsuk's book: *Theory of Shape*, Monografie Matematyczne, Polish Scientific Publishers, Warsaw, 1975.

NOTES

Equitable, Envy-free, and Efficient Cake Cutting for Two People and Its Application to Divisible Goods

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Every child knows how to divide a cake fairly between two children. One child cuts the cake into two pieces, while the other one chooses which piece she wants. If the Cutter does not know the preferences of the Chooser, then the Cutter can guarantee himself only what he perceives to be half of the cake, by cutting the cake into what he believes are two equally valued pieces. In this case, the Chooser receives *at least* half of the cake, usually more, since she will probably not place equal value on the two pieces.

If the Cutter does know the preferences of the Chooser, then he can usually guarantee himself more than half of the cake. As long as their preferences are different, then the Cutter can cut the cake so that the Chooser prefers one piece to the other, while he prefers the piece that she does not select.

This is unlike Sanford's Proof Without Words in this issue [**9**, p. 283], which ensures that each person receives the same amount of frosting and cake. In fact, the cake may not be homogeneous; individual preferences may vary over the cake, as expressed (in this note) by probability density functions. Deciding how to cut a cake under different notions of fairness has been examined for the last fifty years, including Dubins and Spanier's moving-knife procedure [**6**], which for two players, is comparable to Cut and Choose.

By recasting the Dubins-Spanier procedure geometrically, I demonstrate the existence of an optimal planar cut, proving that the optimal planar cut allows two players to "have their cake and eat it, too," as the optimal planar-cut allocation satisfies three important criteria: As in Austin's moving-knife procedure [**1**], both players believe that they receive the same amount of cake, according to their preferences, that is, the allocation is *equitable*; using two planar cuts, Austin's procedure guarantees that both players believe that they receive *exactly* half of the cake. I propose a solution that is not only equitable, but efficient with respect to a single planar cut and envy-free as well. Because there is no other division by a single planar cut that gives both players more of the cake, under their preferences, the solution is *efficient with respect to a planar cut*. The solution is *envy-free* because neither player prefers or envies the piece of cake received by the other player. Equitability can be viewed as second order envy. An allocation is equitable when neither player envies the other player's perceived value of the cake received.

Although the optimal planar-cut result requires that there are only two players, it provides insight into Brams and Taylor's Adjusted Winner procedure [**3**]. Their procedure yields an envy-free, equitable, and efficient division of a set of k divisible goods between two players. For a set of k divisible goods, the planar-cut result yields an alternate, more geometrical proof of the Adjusted Winner procedure, thereby linking cake cutting to the allocation of divisible goods. I relate the optimal planar-cut result to the cake-cutting literature for $n > 2$ players at the end of this note.

A review of the Dubins-Spanier procedure Typically, moving-knife procedures require that a knife move from left to right along an axis, only making cuts perpendicular to the axis. For ease of viewing the procedure, as in Austin [**1**], the cake is pictured as a rectangle in the plane, as if viewing a three-dimensional cake from above. The axis is parallel to one of the sides of the rectangle, as pictured in FIGURE 1. For two players, the Dubins-Spanier procedure mimics Cut and Choose, as follows:

> A referee holds a knife at the left edge of the cake and moves it slowly across the cake so that, at every point along the horizontal edge, the knife remains parallel to its starting position at the left edge (see FIGURE 1a). At any time, either player can call "cut." When this occurs, the player who called "cut" receives the piece to the left of the knife, and the other player receives the piece to the right of the knife.

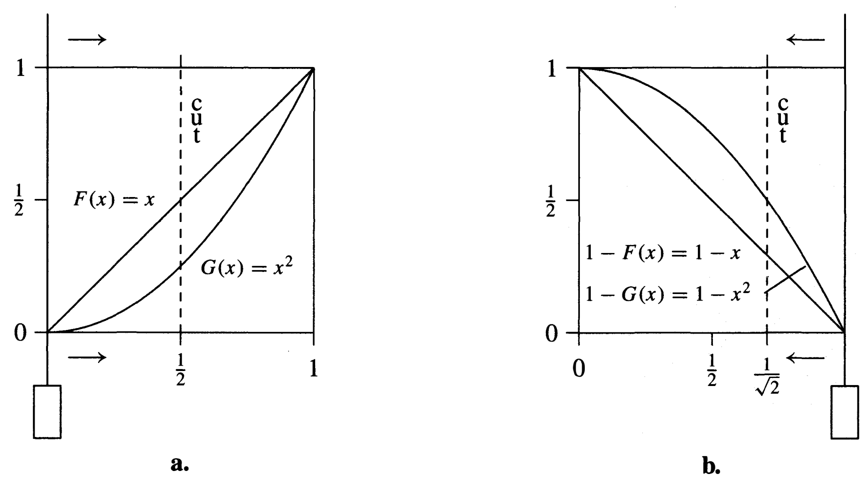

Figure 1 The Dubins-Spanier procedure with the knife moving from left to right (**a**) and from right to left (**b**)

A player can ensure that he receives at least half of the cake by calling "cut" when the knife reaches the halfway point, according to his preferences. If a player waits to call "cut" after the knife has passed *his* halfway point, then his opponent may call "cut" first. In this event, the player who waited receives the piece on the right that he perceives to be less than half of the cake.

The Dubins-Spanier procedure treats the players symmetrically, unlike Cut and Choose, which designates one player as the Cutter. However, in the Dubins-Spanier procedure, the role of Cutter is determined by the axis and the direction of the knife. Indeed, the knife could move from right to left. If both players adhere to the strategy that ensures they receive at least half of the cake, then the player who calls "cut" first when the knife moves from left to right is not the player who calls "cut" when the knife moves from right to left. Further, the axis may be oriented in an infinite number of ways that would change which player would call "cut" and how much the other player would receive. The following example demonstrates how the outcome of the Dubins-Spanier procedure depends on the direction of the knife.

EXAMPLE 1. Switching the knife's direction in the Dubins-Spanier procedure

Suppose that two players, A and B, have preferences over a cake that is represented by the unit square. Further, let player A's preference for the cake be represented by the probability density function $f(x, y) = 1$. The cumulative distribution function (c.d.f.) of f with respect to x is $F(x) = \int_0^x \int_0^1 1\, dy\, dx = x$. Let player B's preference for the cake be represented by the probability measure $g(x, y) = 2x$, indicating that B despises the left edge of the cake and greatly prefers the right edge. The c.d.f. of g with respect to x is $G(x) = x^2$. Both F and G are graphed in FIGURE 1a.

Suppose players A and B use the Dubins-Spanier procedure and the strategies that ensure that each receives at least half of the cake. If the knife proceeds from left to right, as in FIGURE 1a, then player A calls "cut" when $x = 1/2$ because $F(1/2) = 1/2$. Player A receives the left piece, which he values at $1/2$. Player B receives the right piece, which she values at $1 - G(1/2) = 3/4$.

To view the Dubins-Spanier procedure more easily when the knife proceeds from right to left, $1 - F$ and $1 - G$ are graphed in FIGURE 1b. Player B calls "cut" at $x = 1/\sqrt{2}$ because $G(1/\sqrt{2}) = 1/2$; she receives the right piece, which she values at $1/2$. Player A receives the left piece, which he values at $1 - [1 - F(1/\sqrt{2})] = F(1/\sqrt{2}) = 1/\sqrt{2} \approx 0.7071$. To divide the cake in such a way that both players receive the same value, which is more than $1/2$, the planar cut must occur somewhere between $x = 1/2$ and $x = 1/\sqrt{2}$.

The optimal planar cut Adapting the two approaches described in the example provides an allocation that is equitable, envy-free, and efficient with respect to a single planar cut.

EXAMPLE 2. Continuing Example 1

For players A and B as described in Example 1, graph $F(x) = x$ and $1 - G(x) = 1 - x^2$ on the same unit square, as in FIGURE 2. These functions intersect at $\beta = (\sqrt{5} - 1)/2 \approx 0.6180$. By cutting the cake at $x = \beta$, either player may choose the piece that he or she wants. Player A chooses the left piece that he values at $\alpha = \beta >$

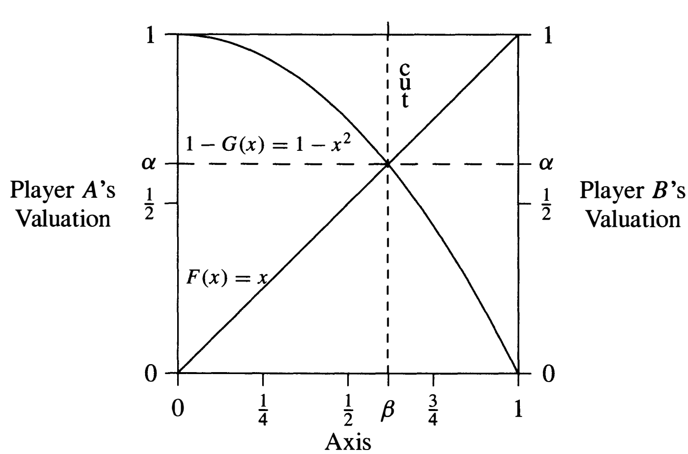

Figure 2 Using cumulative distribution functions to determine where to cut the cake. Here, $F(\beta) = 1 - G(\beta) = \alpha$.

$1/2$. And, player B chooses the right piece that she values at α because $\alpha = F(\beta) = 1 - G(\beta) > 1/2$.

Restricting to a single cut, perpendicular to this axis, cutting the cake at $x = \beta$ yields an equitable, envy-free, and efficient allocation. Both players receive pieces that they value at α. Notice that any other planar cut perpendicular to the x-axis gives a lesser amount to one of the players; if the cake is cut elsewhere, then F and $1 - G$, and G and $1 - F$, cannot both increase.

In the examples, the cake was represented by a two-dimensional compact set. As the knife moved along, it was held perpendicular to a fixed axis or line in the plane. Assume that the cake is any three-dimensional compact set; the cake does not need to be convex, or even connected.

Let the orientation of the axis be determined by a three-dimensional unit vector. The set of all possible axes is the unit sphere in $\mathbb{R}^3$. A best axis is proved to exist in the following theorem. Assume that the players' preferences are given by absolutely continuous probability measures.

THEOREM (Optimal Planar Cut). *There exists a division of cake between two players that is equitable, envy-free, and efficient with respect to a single planar cut.*

Proof. For a fixed unit vector $\mathbf{v}$, define the axis to be the line through the origin parallel to $\mathbf{v}$. Construct probability density functions over the axis by projecting each player's measures onto the axis. Each point on the axis receives the measure of the region of the cake in the plane through the point that has $\mathbf{v}$ as its normal vector. Let $F_{\mathbf{v}}(x)$ and $G_{\mathbf{v}}(x)$ be the cumulative density functions for players 1 and 2, respectively, induced by the p.d.f.s for the axis given by $\mathbf{v}$. Since the measures are absolutely continuous, it follows that the players' c.d.f.s are continuous.

Both $F_{\mathbf{v}}$ and $G_{\mathbf{v}}$ are monotonically nondecreasing, continuous functions in x. Consequently, $1 - G_{\mathbf{v}}$ is a monotonically nonincreasing, continuous function. Therefore, $F_{\mathbf{v}}$ and $1 - G_{\mathbf{v}}$ intersect at an $x_{\mathbf{v}}$. If the players cut the cake at $x_{\mathbf{v}}$ and select their pieces rationally, then they receive a piece of cake valued at $\alpha_{\mathbf{v}}$.

Every axis $\mathbf{v}$ produces a value $\alpha_{\mathbf{v}}$, where $\mathbf{v}$ and $-\mathbf{v}$ yield the same value. The map from the compact set of axes to the set of values is continuous, since the players' preferences are absolutely continuous. This implies that there exists an axis $\mathbf{v}^*$ such that $\alpha_{\mathbf{v}^*}$ is a maximum. Using the axis $\mathbf{v}^*$ yields an allocation that is equitable, envy-free, and efficient with respect to a planar cut. ∎

To view the above theorem as a process, both players would submit their probability measures simultaneously and use the above theorem to determine where to cut the cake. As in Cut and Choose, one player's knowledge of his opponent's preferences can lead to the exploitation of this information by determining an optimal response to the opponent's preferences. Not only do the players have to factor in the information that they know about their opponents, but they also have to factor in the information that they *know* their opponents know; this can go on indefinitely.

To eliminate such scheming, the players could submit their measures and then choose an axis randomly. The outcome may not be efficient over all axes, but it would be equitable, envy-free, and efficient for cuts perpendicular to the fixed axis. Yet, only by submitting sincere probability measures can the players ensure that they receive *at least* half of the cake. Compare this to Cut and Choose where the Cutter can guarantee himself to receive what he perceives to be *exactly* half of the cake.

Although it is not realistic for players to know their p.d.f.s over the cake, limiting the optimal planar-cut result to one of existence, the theorem can be used to allocate a finite set of divisible goods. Both cake cutting and fair division procedures are often taught in mathematics courses for nonmajors; indeed, COMAP **[5]** considers cake cutting and a fair division procedure called the Adjusted Winner procedure. The optimal planar-cut theorem geometrically links these two topics by piecing the discrete set of goods together to form a cake.

Allocating a set of k divisible goods Brams and Taylor [3] develop the Adjusted Winner (AW) procedure to allocate a discrete set of k divisible goods $(G_1, \ldots, G_k)$ between two players, A and B. The AW procedure requires that each player (simultaneously and independently) distribute 100 points over the goods to reflect the relative worth of the items to the player where all goods receive positive values. It is more convenient to assume that they distribute fractions of one point over the items, again with the positivity constraint. Suppose players A and B place values a_i and b_i, respectively, on good G_i, so that $\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i = 1$, and both a_i and b_i are greater than zero for all i. The following results can be modified to allow some of the a_is or b_is to be zero.

The AW procedure requires that each good be divisible and that the players value a percentage of a good at the percentage that they value the entire good. Hence, half of a good valued at v yields $v/2$; this implies that the p.d.f. is uniform over each item. Further, preferences are such that the value received from more than one good is additive: if the value attained from G_1 and G_2 is v_1 and v_2, respectively, then the value achieved from receiving both G_1 and G_2 is the sum $v_1 + v_2$.

Define an allocation as a vector $\mathbf{p} = (p_1, \ldots, p_k)$ such that A receives p_i of good i and B receives $\bar{p}_i = 1 - p_i$ of good i. Player A values this allocation at $\mathbf{p} \cdot \mathbf{a}$. Similarly, B values this allocation at $\bar{\mathbf{p}} \cdot \mathbf{b}$. The AW procedure returns an equitable, envy-free, and efficient allocation that is efficient when compared to *all* other feasible allocations. Hence, there is no other allocation that is better for both players. Realize that efficiency in this case is not restricted to planar cuts. Before we can apply the Optimal Planar-Cut Theorem, we need a cake!

The following theorem and lemma indicate that an equitable, envy-free, and efficient allocation exists that divides at most one good between the two players. This is crucial for constructing the cake. Once the cake is constructed, the Optimal Planar-Cut Theorem yields an equitable, envy-free, and efficient allocation that divides at most one good. Even though the solution is determined by the Optimal Planar-Cut Theorem, the solution's efficiency is not restricted to planar cuts.

THEOREM (Existence). *There exists an allocation of a set of k divisible goods between two players that is equitable, envy-free, and efficient.*

Proof. As before, assume that the players place positive weight on each good. An equitable, envy-free, and efficient allocation can be determined by maximizing $\mathbf{p} \cdot \mathbf{a}$ subject to the constraints that $0 \leq p_i \leq 1$ for all i, and $\mathbf{p} \cdot \mathbf{a} = \bar{\mathbf{p}} \cdot \mathbf{b}$. Realize that $\mathbf{p} \cdot \mathbf{a}$ is a continuous function with respect to changes in $\mathbf{p}$. The set of feasible allocation vectors $\mathbf{p}$ such that $0 \leq p_i \leq 1$ for all i forms a compact set, namely the n-dimensional cube. The set of vectors $\mathbf{p}$ that satisfy the additional restriction that $\mathbf{p} \cdot \mathbf{a} = \bar{\mathbf{p}} \cdot \mathbf{b}$ is a closed set. The intersection of these two sets is compact and nonempty; it is nonempty because the vector $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is in both sets.

There exists an allocation that maximizes $\mathbf{p} \cdot \mathbf{a}$ subject to the constraints, since a continuous function achieves a maximum on a compact set. This allocation will be equitable (due to the constraints) and envy-free (since both players prefer their allocations to the other players' allocation). The allocation is efficient because if there exists an allocation that is better for both players, then continuity implies that there exists an allocation that is better by the same amount for both players. But this would contradict the solution to the optimization problem. Hence, an equitable, envy-free, and efficient allocation exists. ■

To use our cake cutting result to find such an allocation, it is necessary to consider how to arrange the goods into a single cake. The following lemma shows that there exists an equitable, envy-free, and efficient allocation that only divides at most one of the k goods between the two players (the other goods are given entirely to one of the two players). This allows us to construct a cake.

LEMMA. *There exists an equitable, envy-free, and efficient allocation that only divides at most one of the k goods.*

Proof. Let $\mathbf{p}$ be an equitable, envy-free, and efficient allocation that divides two or more goods. Suppose G_i and G_j are divided between A and B such that A receives p_i of G_i and p_j of G_j (and $p_i, p_j \in (0, 1)$). Without loss of generality, assume that i and j are numbered such that $a_i/b_i \geq a_j/b_j$ or equivalently, $a_i/a_j \geq b_i/b_j$.

If $a_i/a_j > b_i/b_j$, then we modify $\mathbf{p}$ to create an allocation that is equitable and envy-free, but valued more than $\mathbf{p}$. This contradicts the efficiency of $\mathbf{p}$. The adjusted allocation assigns $p_i^* = p_i + \epsilon$ of G_i to A and $p_j^* = p_j - \delta$ of G_j to A where ϵ and δ are defined by

$$\epsilon = \min\{1 - p_i, p_j\} \quad \text{and} \quad \delta = \left(\frac{a_i + b_i}{a_j + b_j}\right)\epsilon \quad \text{when} \quad \frac{a_i + b_i}{a_j + b_j} \leq 1, \text{ and}$$

$$\delta = \min\{1 - p_i, p_j\} \quad \text{and} \quad \epsilon = \left(\frac{a_j + b_j}{a_i + b_i}\right)\delta \quad \text{when} \quad \frac{a_i + b_i}{a_j + b_j} > 1.$$

The definition of ϵ and δ guarantees that p_i^* and p_j^* are in $[0, 1]$. Player A values this new allocation at $\mathbf{p} \cdot \mathbf{a} + \epsilon a_i - \delta a_j$ while B values her share at $\bar{\mathbf{p}} \cdot \mathbf{b} - \epsilon b_i + \delta b_j$. Since the initial allocation was equitable by assumption (that is, $\mathbf{p} \cdot \mathbf{a} = \bar{\mathbf{p}} \cdot \mathbf{b}$), the definition of ϵ and δ ensure equitability. However, this new allocation is valued more than $\mathbf{p}$ since the new allocation differs in value to $\mathbf{p}$ by $\epsilon a_i - \delta a_j$, which is positive. This follows since

$$\frac{a_i}{a_j} > \frac{(a_i + b_i)/2}{(a_j + b_j)/2} = \frac{\delta}{\epsilon} > \frac{b_i}{b_j}.$$

We have our contradiction.

If $a_i/a_j = b_i/b_j$, then we modify $\mathbf{p}$ to create an allocation that is equitable and envy-free and still efficient, valued the same as $\mathbf{p}$. However, the adjusted allocation divides fewer goods. This can be repeated to arrive at an allocation that divides only one good. Let v_A be the value that A receives from G_i and G_j under $\mathbf{p}$, that is, $v_A = p_i a_i + p_j a_j$. Similarly, define $v_B = (1 - p_i)b_i + (1 - p_j)b_j$. At least one of the following inequalities holds: $a_i \geq v_A, a_j \geq v_A, b_i \geq v_B$, or $b_j \geq v_B$. The contrary implies that $v_A = p_i a_i + p_j a_j < p_i v_A + p_j v_A$ and $v_B = (1 - p_i)b_i + (1 - p_j)b_j < (1 - p_i)v_B + (1 - p_j)v_B$. These two inequalities cannot be true since either $p_i + p_j$ or $2 - (p_i + p_j)$ is greater than or equal to 1. Without loss of generality, assume $a_i \geq v_A$. Define $p_i^* = v_A/a_i$ and $p_j^* = 0$. The new allocation is equally valued by A and B as $\mathbf{p}$ is, but gives G_j to player B. ■

Since there exists an equitable, envy-free, and efficient allocation that divides at most one good, we can arrange the goods from left to right and view the cake as a line segment (with a single axis, as in Example 2). Any left-right ordering and the additive and divisible nature of the goods mean that we can create a piecewise-linear c.d.f. that represents the preferences over the goods. Define a p.d.f. for player A over the interval

$[0, k]$ by: $f(x) = a_i$ on $[i - 1, i]$ for $i = 1$ to k. This p.d.f. defines a piecewise-linear c.d.f. given by F. Define g and G similarly for player B. The Optimal Planar-Cut Theorem can be applied to the interval $[0, k]$, thereby dividing the goods, ensuring envy-freedom and equitability. Assume F and $1 - G$ intersect in $[i - 1, i]$, then one player receives G_1 through G_{i-1} and a portion of G_i while the other player receives G_{i+1} through G_k and the other portion of G_i. However, efficiency holds only for this particular left-right ordering.

This idea of representing goods or items as a cake or contiguous object to be divided is commonly used in cake cutting. Indeed, Peterson and Su [**7**] represent a set of chores as a cake, assuming, reasonably, that all chores can be infinitely divisible (for instance, cutting any sized portion of a lawn). The cake-cutting literature focuses on the existence of divisions satisfying certain properties and on the existence of implementable algorithms. For this reason, it is rare to explicitly construct a cake out of goods, ordering the objects, defining p.d.f.s, etc. Just as the Optimal Planar-Cut Theorem considers all possible axes for a three-dimensional cake, we should consider all left-right orderings of the goods to determine the ordering that yields the greatest equitable, envy-free allocation. This could require constructing $k!/2$ p.d.f.s and c.d.f.s (due to symmetry). However, one ordering is ideal and yields the equitable, envy-free, and efficient solution achieved by the AW procedure.

After the players distribute the values a_i and b_i, re-number the goods such that

$$\frac{a_1}{b_1} \geq \frac{a_2}{b_2} \geq \cdots \geq \frac{a_k}{b_k},$$

in decreasing order of the ratio of A's value to B's value. Call this the *Descending Ratios of Comparative Worth* or *DRCW* ordering. If G_i is shared by A and B under the DRCW ordering, then another ordering is equivalent to DRCW if it permutes the first $i - 1$ goods and the last $k - i$ goods or permutes good i with a good j such that $a_i/b_i = a_j/b_j$.

PROPOSITION. *Under the DRCW ordering, the Optimal Planar-Cut Theorem yields an equitable, envy-free, and efficient allocation of k divisible goods between two players.*

Proof. The Optimal Planar-Cut Theorem yields an envy-free and equitable allocation for any ordering. The DRCW ordering guarantees that the allocation is efficient and that player A will prefer the left piece while player B will prefer the right piece. Indeed, assume that there exists an envy-free, equitable, and efficient allocation that comes from an ordering that is not equivalent to the DRCW ordering. Hence, there exist goods G_i and G_j such that $a_i/b_i > a_j/b_j$, where A gets all or a portion of G_j and B gets all or a portion of G_i. The proof of the Lemma can be used to construct an allocation that is preferred by *both* players. Hence, A never receives a portion of a good G_j and doesn't receive a portion of a good G_i, where $a_i/b_i > a_j/b_j$, or we have a contradiction. Therefore, the DRCW ordering yields an equitable, envy-free, and efficient allocation. ■

A proof that AW yields an equitable, envy-free, and efficient allocation appears in Brams and Taylor [**3**, pp. 85–88]. They also require placing the goods in the DRCW ordering. The AW procedure adjusts an initial efficient allocation through an "equity adjustment" that guarantees equitability. This sometimes requires adjusting the initial allocation by giving an entire good to one player when it was initially allocated to the other. This slight complication is avoided in the geometrical approach as the application of the Optimal Planar-Cut Theorem carries out the equity adjustment. The following example demonstrates the application of the Optimal Planar-Cut Theorem to the divisible good case.

EXAMPLE 3. An allocation of divisible goods

Suppose that player A values G_1 through G_4 at 0.3, 0.35, 0.1, and 0.25. Further, suppose that player B values G_1 through G_4 at 0.2, 0.3, 0.1, and 0.4. Notice that the goods satisfy the DRCW ordering. Let f and g be the associated p.d.f.s for players A and B, respectively. Define F and G by

$$F(x) = \begin{cases} 0.30x \\ 0.35(x-1)+0.30 \\ 0.10(x-2)+0.65 \\ 0.25(x-3)+0.75 \end{cases} \text{ and } G(x) = \begin{cases} 0.2x & \text{for } x \in [0,1] \\ 0.3(x-1)+0.2 & \text{for } x \in [1,2] \\ 0.1(x-2)+0.5 & \text{for } x \in [2,3] \\ 0.4(x-3)+0.6 & \text{for } x \in [3,4]. \end{cases}$$

The graphs of F and $1-G$ intersect at $\beta \in [1,2]$ (FIGURE 3), where β satisfies

$$F(\beta) = 0.35(\beta-1)+0.3 = 1-[0.3(\beta-1)+0.2] = 1-G(\beta).$$

Cutting the cake at $\beta = 115/65$ awards G_1 and $10/13$ of G_2 to player A and G_3, G_4, and $3/13$ of G_2 to player B. Both players A and B value their allocations at $\alpha = F(\beta) = 1-G(\beta) = 74/130 \approx 0.5923$.

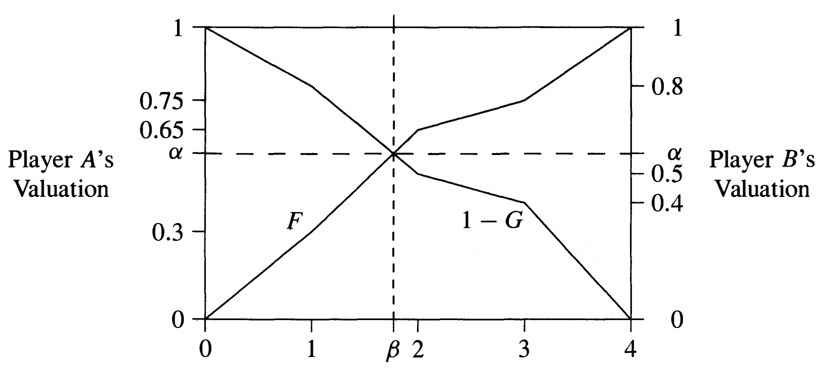

Figure 3 F and $1-G$ intersect at $\beta = 115/65$. Player A values his piece (left) at $F(\beta) = \alpha = 74/130$. Player B values her piece (right) at $1-G(\beta) = \alpha$.

The Optimal Planar-Cut Theorem and AW procedure can be applied to the division of chores, too. If the point value indicates how much a player dislikes a particular chore, then the same construction can be used, except that players select the division that they value less than half of the total. Of course, the Optimal Planar-Cut Theorem and AW only use one cut to divide the items between the two players. In general, in cake cutting and chore division, the cuts are not restricted to being perpendicular to a particular axis and there is no restriction in using only $n-1$ cuts to divide the cake into n pieces for n people.

For n players, the Dubins-Spanier procedure guarantees that each player receives at least $1/n$ of the cake with $n-1$ cuts, but it does not guarantee envy-freedom or equitability. Stromquist **[10]** provides a moving-knife procedure for three players that guarantees envy-freedom with the minimal number of two cuts. He also provides an existence proof of an n-player, envy-free allocation with $n-1$ cuts, as does Woodall **[11]**. Brams, Taylor, and Zwicker **[4]** derive a moving-knife procedure for four players that yields an envy-free division with eleven cuts. Brams and Taylor **[2]** describe a protocol for envy-free division for n players that may require an unbounded number of cuts. Recently, Peterson and Su **[7]** provided a procedure for four-person envy-free chore division with a bounded number of cuts. Brams and Taylor [**2**] and Robertson and Webb [**8**] offer accessible overviews of the cake cutting literature.

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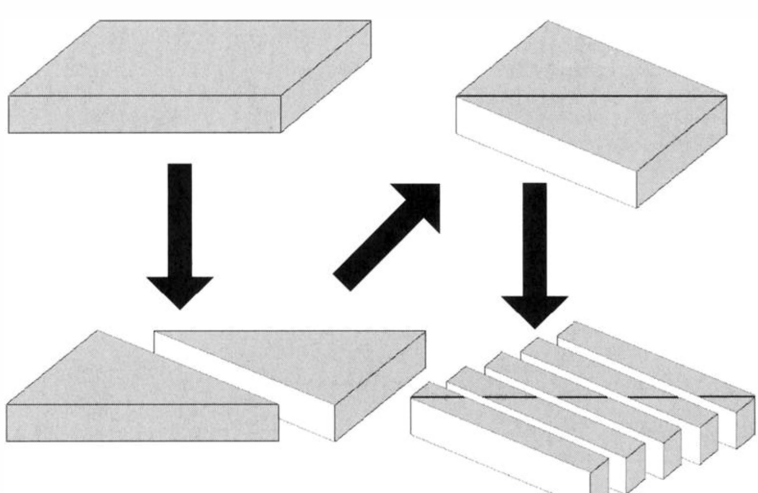
Proof Without Words: Dividing a Frosted Cake



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Semidirect Products: $x \mapsto ax + b$ as a First Example

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After precalculus, mathematics students often leave behind the familiar family of transformations, $x \mapsto ax + b$. We will show that this family, when examined in the right light, leads us to some interesting and important ideas in group theory.

By building on this accessible example, it is possible to introduce the semidirect product (a topic usually first seen at the graduate level) in an undergraduate abstract algebra course. After being introduced to the semidirect product, students are able to better understand the structures of some of the groups of small order that are typically discussed in a first course in abstract algebra.

Constructions Early in any study of abstract algebra, we learn how to construct new groups by taking the (external) direct product of two groups. For this construction, we need not put any restrictions on the operations of the groups we are combining because the group operation for the direct product is defined componentwise using the operations of the factors.

Here is an example that we will later use for comparison. Let $\mathbb{R}^+$ denote the additive group of the real numbers $\mathbb{R}$, and let $\mathbb{R}^\times$ denote the multiplicative group of the nonzero elements of $\mathbb{R}$. Consider the external direct product $\mathbb{R}^+ \times \mathbb{R}^\times$. The group operation on the subgroup $\{(b, 1)\}$ looks like addition

$$(d, 1)(b, 1) = (d + b, 1),$$

and the group operation on the subgroup $\{(0, a)\}$ looks like multiplication

$$(0, c)(0, a) = (0, ca),$$

but these operations merge into one operation for $\mathbb{R}^+ \times \mathbb{R}^\times$

$$(d, c)(b, a) = (d + b, ca).$$

Notice that $\mathbb{R}^+ \times \mathbb{R}^\times$ is abelian because both $\mathbb{R}^+$ and $\mathbb{R}^\times$ are abelian and the operation is defined componentwise. It is easy to show that the external direct product $G = G_1 \times G_2$ has the following properties: If $H = \{(h, 1_{G_2})\}$ and $K = \{(1_{G_1}, k)\}$, then $H \approx G_1$ and $K \approx G_2$ (where we denote isomorphism by $\approx$). Furthermore, $HK = G$; $H \cap K = \{(1_{G_1}, 1_{G_2})\}$, the identity in G; and H and K are normal in G.

More interesting than constructing arbitrary external direct products is discovering which groups are internal direct products of two (or more) of their subgroups. Usually we encounter this early—at order four—when we notice that the Klein four-group $V_4 \approx Z_2 \times Z_2$. Later we learn in the fundamental theorem of finite abelian groups that every finite abelian group is the direct product of cyclic groups.

Another, more subtle way to construct groups is by the semidirect product. This interesting structure is not usually examined in undergraduate abstract algebra texts, although some of the groups we encounter most often in a typical undergraduate abstract algebra course are semidirect products—D_{2n}, the dihedral group of order $2n$ (which consists of the symmetries of a regular n-gon), and S_n, the symmetric group of degree n (which consists of the permutations of $\{1, 2, \ldots n\}$). There is one notable exception; Goodman **[8]** has a section discussing the semidirect product. Hungerford **[11]** and Saracino **[14]** relegate the semidirect product to exercises: Hungerford defines the internal semidirect product and has an exercise to show that each of S_3, D_4, and S_4 is a semidirect product of two of its subgroups, and Saracino defines the external semidirect product and uses it in an exercise to show that if p and q are primes and p divides $q - 1$, then there exists a nonabelian group of order pq.

It is certainly not necessary to wait until graduate school to encounter the semidirect product. We can take the first step toward it by carefully exploring $x \mapsto ax + b$.

Internal semidirect products Consider the vector space $\mathbb{R}^1$ and the transformations

$$x \mapsto ax + b.$$

We will require $a \neq 0$, without which this mapping is not a transformation. Also, notice that if $b \neq 0$, then this is not a linear transformation of $\mathbb{R}^1$ (because 0 is not fixed). Let G be the set of transformations $\{x \mapsto ax + b\}$. We will show that this is a group, where the operation is composition of transformations. First, check that we have closure under composition: if we take two maps, say, $T_{a,b} : x \mapsto ax + b$ and $T_{c,d} : x \mapsto cx + d$ and compose them, we get $T_{c,d} \circ T_{a,b} : x \mapsto cax + (cb + d)$, which is in G, since $ca \neq 0$. Associativity is automatic because composition of arbitrary maps is associative. Taking $a = 1$ and $b = 0$ gives the identity transformation. Finally, we need to check for inverses, and we notice that a condition is required. If $T_{a,b} : x \mapsto ax + b$, then $T_{a,b}^{-1} : x \mapsto x/a - b/a$. Since $a \neq 0$, having a in the denominator causes no problems.

We call the mappings $x \mapsto ax + b$ *affine transformations* and the group G of mappings $x \mapsto ax + b$ with $a \neq 0$ the *affine general linear group* of $\mathbb{R}^1$.

Notice that an affine transformation $T_{a,b} : x \mapsto ax + b$ does two distinct things to x. It scales x by $a \neq 0$, and it translates by b. Clearly, $\mathbb{R}^+$ is isomorphic to the subgroup $N = \{x \mapsto ax + b : a = 1\} = \{x \mapsto x + b\}$, which are the translations; and $\mathbb{R}^\times$ is isomorphic to the subgroup $A = \{x \mapsto ax + b : a \neq 0, b = 0\} = \{x \mapsto ax : a \neq 0\}$, which are the scalings.

The group of affine linear transformations of $\mathbb{R}^1$ is a product of the two subgroups $N = \{x \mapsto x + b\}$ and $A = \{x \mapsto ax : a \neq 0\}$. But, it is not the direct product (we essentially constructed that earlier as $\mathbb{R}^+ \times \mathbb{R}^\times$); it is a semidirect product.

To explore this product, let us examine how the two subgroups N and A sit inside G. First, notice that every affine transformation $T_{a,b} : x \mapsto ax + b : a \neq 0$ is made up by composing an element $T_{1,b}$ from $N = \{x \mapsto x + b\}$ with an element $T_{a,0}$ from $A = \{x \mapsto ax : a \neq 0\}$. Next, we notice that $N \cap A = \{x \mapsto x\} = \{T_{1,0}\}$, the identity transformation. Finally, notice that N is normal in G, denoted $N \lhd G$ (because if $T_{1,d} \in N$ and $T_{a,b} \in G$, then $T_{a,b} \circ T_{1,d} \circ T_{a,b}^{-1} : x \mapsto x + ad \in N$). These three properties expressing how N and A sit inside G characterize the structure called an internal semidirect product.

DEFINITION. A group G is the *internal semidirect product* of N by A, which we denote by $G = N \rtimes A$, when G contains subgroups N and A such that $NA = G$, $N \cap A = \{1\}$, and $N \lhd G$.

Notice that the internal semidirect product is a generalization of the internal direct product, where we require that *both* subgroups be normal in G. The symmetry of N and A in the definition of the direct product $N \times A$ is reflected in the use of the symmetric symbol $\times$. The symbol for the semidirect product $\rtimes$ is asymmetric to remind us that the two subgroups do not sit symmetrically in the semidirect product; the symbol points to the left to remind us of which subgroup is normal.

The normality of N enables the manipulation

$$(n_1a_1)(n_2a_2) = (n_1a_1)(n_1(a_1^{-1}a_1)a_2) = (n_1(a_1n_2a_1^{-1}))(a_1a_2) \in NA,$$

which verifies the closure of the operation in $G = NA$.

Let us consider some familiar groups that are semidirect products.

The smallest order nonabelian group that we meet has order six—D_6 the symmetries of an equilateral triangle. Now, there are two groups of order six—D_6 and Z_6. (Z_6 denotes the cylic group of order 6.) Each has a (normal) subgroup of order three, but Z_6 has only one subgroup of order two (hence, it is normal) while D_6 has three subgroups of order two (none of which is normal). $Z_6 \approx Z_2 \times Z_3$, but $D_6 \approx N \rtimes A$ where N is the subgroup of order three (the rotations) and A is one of the subgroups of order two (one of the subgroups generated by a reflection). In general, D_{2n}, the symmetries of a regular n-gon, is the semidirect product of its rotation subgroup by one of the subgroups generated by a reflection.

Recall that $S_3 \approx D_6$. The alternating group A_3, which consists of the even permutations corresponds to the subgroup of rotations of D_6. It is easy to see in general that S_n is the internal semidirect product of A_n by a subgroup of order two generated by one of the transpositions or two-cycles. (Gallian [7, p. 183] gives a construction of D_{12} as an internal direct product.)

Another semidirect product that students may have seen is the group of Euclidean isometries of the plane or of space. Each of these transformation groups is the semidirect product of the subgroup of translations by the subgroup of isometries that fix the origin. An interesting feature of the spatial case is that the second factor, namely the isometries of the sphere, is a noncommutative group.

Finally, using the internal semidirect product, we can generalize the definition of the affine general linear group of $\mathbb{R}^1$ to $\mathbb{R}^n$ (or any finite dimensional vector space V—we do this near the end of this paper). Let A be the group of all nonsingular (that is, invertible) linear transformations of $\mathbb{R}^n$; this group is called the general linear group of $\mathbb{R}^n$ and is denoted by $\mathrm{GL}(\mathbb{R}^n)$. Consider transformations $x \mapsto a(x) + b$ where $a \in A$ and b is an n-dimensional vector of $\mathbb{R}^n$. Notice that for $\mathbb{R}^1$, A is isomorphic to $\{x \mapsto ax : a \neq 0\}$. We define the *affine general linear group* of $\mathbb{R}^n$ by $\mathrm{AGL}(\mathbb{R}^n) = \{x \mapsto a(x) + b : a \in \mathrm{GL}(\mathbb{R}^n), b \in \mathbb{R}^n\}$. Notice that the affine general linear group of $\mathbb{R}^n$ is the internal semidirect product of N by A where N denotes the normal subgroup of transformations $\{x \mapsto x + b\}$ of $\mathbb{R}^n$.

External semidirect products It is easy to construct external direct products because the operations of the factors do not really merge. The external semidirect product is another matter. We need a clue to suggest to us when we can merge the operations of two unrelated groups into one operation in the semidirect product. Again, we carefully examine $x \mapsto ax + b$ for a clue.

Recall that when we examined the affine general linear group of $\mathbb{R}^1$, we saw that $N \approx \mathbb{R}^+$ and $A \approx \mathbb{R}^\times$ do not play equal roles in the internal semidirect product $G = N \rtimes A$; N is a normal subgroup. Consider conjugation of N by elements of A. Let $T_{1,d} \in N$ and let $T_{a,0} \in A$. Then $T_{a,0} \circ T_{1,d} \circ T_{a,0}^{-1} : x \mapsto x + ad \in N$. Notice the ad, but rather than thinking of a and d as elements of $\mathbb{R}$ and ad as a product in $\mathbb{R}$, think of

$a \in \mathbb{R}^\times$ and $d \in \mathbb{R}^+$ and a acting on $\mathbb{R}^+$. Recall that any $a \in \mathbb{R}^\times$ acts on $\mathbb{R}^+$ as a group automorphism $a(d) = ad$. (We should not be scared of the term "acts on;" it is simply the modern substitute for "permutes." And the term *automorphism* is just a shorthand for an isomorphism from the group to itself.)

Our example suggests that we can merge operations for an external semidirect product if there is an action of one of the groups on the other.

Here, finally, is the definition of the external semidirect product.

DEFINITION. Let X and A be groups, and let θ be a given action of A on X; that is, a homomorphism $\theta : A \to \text{Aut}(X)$ (where Aut(X) denotes the group of automorphisms of X). Then, for $c \in A, \theta(c) : X \to X$, and if $b \in X$, we denote its image under this automorphism by $\theta(c)(b)$. The *external semidirect product* $X \rtimes_\theta A$ *of the group* X *and the group* A *relative to* θ is, as a set, simply $X \times A$. We make this into a group by defining $(d, c)(b, a) = (d\theta(c)(b), ca)$. (When θ is clear, we will write $X \rtimes A$.)

The action θ becomes conjugation in the semidirect product, that is, $(\theta(a)(b), 1) = (1, a)(b, 1)(1, a)^{-1}$.

Let $\overline{X} = \{(b, 1) : b \in X\}$, and let $\overline{A} = \{(1, a) : a \in A\}$. Then $X \rtimes_\theta A$ is easily seen to be the internal semidirect product of $\overline{X}$ by $\overline{A}$.

If θ is trivial, that is, if the image of $\theta = 1$, then $X \rtimes_\theta A$ is simply the external direct product $X \times A$.

We now construct external semidirect products that are isomorphic to the two affine general linear groups we mentioned above.

Because each $c \in \mathbb{R}^\times$ acts on $\mathbb{R}^+$ as a group automorphism $c(b) = cb$, we can construct $\mathbb{R}^+ \rtimes \mathbb{R}^\times$. The elements of our group are the elements of the Cartesian product $\mathbb{R}^+ \times \mathbb{R}^\times$, and the group operation is $(d, c)(b, a) = (cb + d, ca)$. This group is isomorphic to the affine general linear group of $\mathbb{R}^1$. Notice that although $\mathbb{R}^+$ and $\mathbb{R}^\times$ are abelian, $\mathbb{R}^+ \rtimes \mathbb{R}^\times$ is not abelian (unlike $\mathbb{R}^+ \times \mathbb{R}^\times$).

Because $\text{GL}(\mathbb{R}^n)$ acts on $\mathbb{R}^n$, we can construct $\mathbb{R}^n \rtimes \text{GL}(\mathbb{R}^n)$. The elements of our group are the elements of the Cartesian product $\mathbb{R}^n \times \text{GL}(\mathbb{R}^n) = \{(b, a) : b \in \mathbb{R}^n, a \in \text{GL}(\mathbb{R}^n)\}$, and the group operation is $(d, c)(b, a) = (c(b) + d, ca)$. This group is isomorphic to $\text{AGL}(\mathbb{R}^n)$.

Holomorphs and Cayley's theorem The transformation $x \mapsto ax + b$ is also a key to understanding Cayley's theorem. But before looking for this representation, we will use the semidirect product to introduce another idea from group theory—the idea of a holomorph. Consider an arbitrary group X, which as usual is written multiplicatively. Let A be a subgroup of $\text{Aut}(X)$. Notice that $\text{Aut}(X)$ is a subgroup of the symmetric group on X, denoted $\text{Sym}(X)$. Recall that the symmetric group on X is the group of all bijections of X; that is, the set of permutations of X. For any $a \in A$ and $b \in X$, let $\Phi_{a,b}$ be the bijection of X under which the value of any $x \in X$ is obtained by multiplying $a(x)$ from the left by b, that is, $\Phi_{a,b}(x) = ba(x)$; note that if X is an additive abelian group then this reduces to the now very familiar $\Phi_{a,b}(x) = a(x) + b$.

We define the *(relative) holomorph* $H(X, A)$ of (X, A) to be the subgroup of $\text{Sym}(X)$ consisting of all $\Phi_{a,b}$ with a varying in A and b varying in X.

Consider the relative holomorph $H(X, 1)$. This is the image in $\text{Sym}(X)$ of the mapping $b \mapsto \Phi_{1,b}, b \in X$. Notice that $\Phi_{1,b}$ is simply the mapping of X into $\text{Sym}(X)$ given by left multiplication by b. This reminds us of Cayley's theorem.

In 1854, Cayley **[4]** stated his theorem that every group of order n is isomorphic to a subgroup of S_n: "A set of symbols, $1, \alpha, \beta, \ldots$ all of them different, and such that the product of any two of them into itself, belongs to the set, is said to be a *group*. It follows that if the entire group is multiplied by any one of the symbols, either as further or nearer factor, the effect is simply to reproduce the group ...."

$H(X, 1)$, obtained from the proof of Cayley's theorem, is called the *left regular representation* of X in $\mathrm{Sym}(X)$. In the case of the additive group $\mathbb{R}^+$, the set of translations $\{\Phi_{1,b}\} = \{x \mapsto x + b\}$ is the regular representation of $\mathbb{R}^+$.

Notice that the left regular representation of X, $H(X, 1)$, is a normal subgroup of $H(X, A)$. (Usually the left regular representation is not a normal subgroup of $\mathrm{Sym}(X)$, but it is a normal subgroup of $H(X, A)$ which is a subgroup of $\mathrm{Sym}(X)$.) Moreover, for any subgroup A of $\mathrm{Aut}(X)$, the relative holomorph $H(X, A)$ is the internal semidirect product of the left regular representation $H(X, 1)$ by A.

We define the *(full) holomorph* $H(X)$ of X by putting $H(X) = H(X, \mathrm{Aut}(X))$. The holomorph of X essentially combines X and its group of automorphisms $\mathrm{Aut}(X)$ to form a larger group.

Affine general linear group By using the idea of holomorph, we can define the affine general linear group of an arbitrary finite dimensional vector space V. Recall that the group of all nonsingular linear transformations of a finite dimensional vector space V over a field k is called the general linear group of V and is denoted by $\mathrm{GL}(V)$. We define the *affine general linear group* $\mathrm{AGL}(V)$ of V by putting $\mathrm{AGL}(V) = H(V, \mathrm{GL}(V))$. In this way, we define $\mathrm{AGL}(V)$ as an internal semidirect product. Also, because $\mathrm{GL}(V)$ is a subgroup of $\mathrm{Aut}(V)$, we may think of $\mathrm{AGL}(V)$ as the external semidirect product $V \rtimes \mathrm{GL}(V)$. Of course, the internal semidirect product and the external semidirect product are isomorphic.

A Theorem of Burnside Abhyankar [**1**] begins with $x \mapsto ax + b$ and proves a theorem of Burnside that a 2-transitive permutation group has a unique minimal normal subgroup and that subgroup is either elementary abelian or nonabelian simple. This theorem, which first appeared in section 134 of the 1897 edition of Burnside's book [**3**] and later in section 154 of the second edition (see also theorem 4.1B and theorem 7.2E of Dixon and Mortimer [**5**]), suffices to settle a portion of Hilbert's Thirteenth Problem and is a cornerstone of the Classification of Doubly Transitive groups (see section 7.7 of Dixon and Mortimer [**5**]). The Classification of Doubly Transitive Groups was the first amazing application of the Classification Theorem of Finite Simple Groups, which Gorenstein called the Enormous Theorem.

A first step to all of these results can be taken by carefully exploring the familiar transformation $x \mapsto ax + b$.

More Of course, more information about the ideas that were mentioned above can be found in standard graduate texts on abstract algebra (Dummit and Foote [**6**] and Hungerford [**11**]) or in advanced texts on the theory of groups [**2, 9, 10, 13**]. A long section containing many examples and exercises about semidirect products can be found in Weinstein [**15**]. Rotman [**13**] discusses the affine general linear group and holomorphs. Holomorphs also were discussed by Burnside [**3**].

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There Are Three Methods for Solving Problems

Method 0
Get the answer from the teacher.

Method 1
Oh no! A problem! Oh no!
I can't do it! I can't!
Oh no! Oh no!

How will I solve this problem?
How will I get the right answer? Oh no!

There must be a method.
I'll find the right one!

How will I find the right method?
What if I find the wrong one!
Oh no!

I might find the right method,
maybe, with luck.

But what if I make a mistake,
even just one?
Oh no! Oh no!

I'll try and I'll try.
I'll think and I'll thunk.

Here's an answer.
It might be the wrong one.
Oh no! Oh no!

Teacher, teacher.
Is my answer correct?

Method 2
That's an interesting puzzle.
Please show me that piece, it looks nice.

That piece over there looks nice too.
I wonder if these two pieces connect.

I'll put this one here and that one nearby.
Let's see where this one goes next.

I could try it right there,
Perhaps it goes here.

This spot is just right.
Now my puzzle is solved.

And I have my answer, too.
That's nice.

My solution is correct.
And my answer's just right!

Now I have three methods, I do.
Which one should I use just now?
I wonder, I do.

Could there be any more methods?
I wonder that too.

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Four Ways to Evaluate a Poisson Integral

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In general, it is difficult to decide whether or not a given function can be integrated in elementary ways. In light of this, it is quite surprising that the value of the Poisson integral

$$I(x) = \int_0^\pi \ln(1 - 2x\cos\theta + x^2)\, d\theta$$

can be determined precisely. Even more surprising is that we can do so for every value of the parameter x. Using four different methods, we will show that

$$I(x) = \begin{cases} 0, & \text{if } |x| < 1; \\ 2\pi \ln|x|, & \text{if } |x| > 1. \end{cases}$$

Our integral is one of several known as the Poisson Integral; all are related in some way to Poisson's integral formula, which recovers an analytic function on the disk from its boundary values, a relationship we mention below. However, none of our methods involves complex analysis at all. The first one uses Riemann sums and relies on a trigonometric identity. The second method is based on a functional equation and involves a sequence of integral substitutions. The third method uses parametric differentiation and the half-angle substitution. We finish with an approach based on infinite series. It is interesting to see how wide a range of mathematical topics are exploited. These evaluations are suitable for an advanced calculus class and provide a very nice application of Riemann sums, functional equations, parametric differentiation, and infinite series.

We begin with three elementary observations:

1. $I(0) = 0$.
2. $I(-x) = I(x)$.
3. $I(x) = 2\pi \ln|x| + I(1/x), \quad (x \neq 0)$.

The reader can probably supply the proofs for these, but we will demonstrate the third. If $x \neq 0$, we have

$$\begin{aligned} I(x) &= \int_0^\pi \ln\left[x^2\left(1 - \frac{2}{x}\cos\theta + \frac{1}{x^2}\right)\right] d\theta \\ &= \int_0^\pi \ln x^2\, d\theta + I(1/x) = 2\pi \ln|x| + I(1/x). \end{aligned}$$

In view of this third observation, our main formula follows easily once we show that $I(x) = 0$ for $|x| < 1$. This will be the goal of the next four sections.

I. Using Riemann sums Since

$$1 - 2x\cos\theta + x^2 \geq (1 - |x|)^2, \quad \text{for } |x| < 1,$$

the integrand is continuous and integrable. Partition the interval $[0, \pi]$ into n equal subintervals by the partition points $x_k = k\pi/n$, for $1 \le k \le n$. The Riemann sum for $I(x)$, $\mathcal{R}_n$, can be simplified using laws of logarithms:

$$\begin{aligned}\mathcal{R}_n &= \frac{\pi}{n}\sum_{k=1}^{n} \ln\left(1 - 2x\cos\left(\frac{k\pi}{n}\right) + x^2\right)\\ &= \frac{\pi}{n}\ln\left[(1+x)^2\prod_{k=1}^{n-1}\left(1 - 2x\cos\left(\frac{k\pi}{n}\right) + x^2\right)\right]. \qquad (1)\end{aligned}$$

To proceed further, let $\omega = \exp(i\pi/n)$. The distinct roots of the polynomial $x^{2n} - 1$ are ω^k for $-n \le k < n$, so

$$x^{2n} - 1 = \prod_{k=-n}^{n-1}(x - \omega^k).$$

Combining the conjugate factors and appealing to De Moivre's theorem, we find

$$\begin{aligned}x^{2n} - 1 &= (x^2 - 1)\prod_{k=1}^{n-1}(x - \omega^k)(x - \omega^{-k})\\ &= (x^2 - 1)\prod_{k=1}^{n-1}\left(1 - 2x\cos\left(\frac{k\pi}{n}\right) + x^2\right),\end{aligned}$$

so that

$$\prod_{k=1}^{n-1}\left(1 - 2x\cos\left(\frac{k\pi}{n}\right) + x^2\right) = \frac{x^{2n} - 1}{x^2 - 1}. \qquad (2)$$

Substituting the identity (2) into (1), we have

$$\mathcal{R}_n = \frac{\pi}{n}\ln\left(\frac{x+1}{x-1}(x^{2n} - 1)\right).$$

Since $|\,x\,| < 1$, $x^{2n} \to 0$ as $n \to \infty$. Hence, we obtain

$$I(x) = \lim_{n\to\infty}\mathcal{R}_n = \lim_{n\to\infty}\frac{\pi}{n}\ln\left(\frac{x+1}{x-1}(x^{2n} - 1)\right) = 0.$$

Remark This method relied on the trigonometric identity (2), which is interesting in its own right.

II. Using a functional equation The functional equation we have in mind is

$$I(x) = I(-x) = \tfrac{1}{2}\,I(x^2). \qquad (3)$$

Adding the two integrals below and using laws of logarithms, we obtain

$$I(x) + I(-x) = \int_0^{\pi}\ln(1 - 2x^2\cos 2\theta + x^4)\,d\theta.$$

Setting $\alpha = 2\theta$ gives

$$I(x) + I(-x) = \frac{1}{2}\int_0^{2\pi} \ln(1 - 2x^2\cos\alpha + x^4)\, d\alpha$$
$$= \frac{1}{2} I(x^2) + \frac{1}{2}\int_{\pi}^{2\pi} \ln(1 - 2x^2\cos\alpha + x^4)\, d\alpha.$$

The substitution $\alpha = 2\pi - t$ in the last integral shows that it is exactly the same as the first integral. Since the two terms on the left are the same (recalling that $I(x) = I(-x)$), we obtain (3).

Applying equation (3) repeatedly, we find that

$$I(x) = \frac{1}{2} I(x^2) = \frac{1}{2^2} I(x^4) = \cdots = \frac{1}{2^n} I(x^{2^n}).$$

Again we assume that $|x| < 1$, so that $x^{2^n} \to 0$ as $n \to \infty$ and consequently

$$I(x) = \lim_{n\to\infty} \frac{1}{2^n} I(x^{2^n}) = 0.$$

Remark Equation (3) holds for any x. In particular, we have that $I(0) = 0$ and $I(\pm 1) = 0$. The latter equation leads to an added bonus:

$$\int_0^{\pi/2} \ln(\sin\theta)\, d\theta = \int_0^{\pi/2} \ln(\cos\theta)\, d\theta = -\frac{\pi}{2}\ln 2,$$

since, for instance,

$$I(1) = \int_0^{\pi} \ln(2 - 2\cos\theta)\, d\theta = 2\pi\ln 2 + 4\int_0^{\pi/2} \ln(\sin\theta)\, d\theta,$$

and similarly for $I(-1)$. These two integrals are improper. To show convergence, for example, using integration by parts, we have

$$\int_0^{\pi/2} \ln(\sin\theta)\, d\theta = \lim_{\epsilon\to 0} \epsilon\ln(\sin\epsilon) - \lim_{\epsilon\to 0}\int_{\epsilon}^{\pi/2} \frac{\theta\cos\theta}{\sin\theta}\, d\theta$$
$$= -\int_0^{\pi/2} \theta\cot\theta\, d\theta.$$

Since $\theta\cot\theta$ is Riemann integrable on $[0, \pi/2]$, $\int_0^{\pi/2} \ln(\sin\theta)\, d\theta$ converges.

III. Using parametric differentiation Since $I(x)$ is differentiable for $|x| < 1$, we apply the Leibniz rule to $I(x)$ to find

$$I'(x) = \int_0^{\pi} \frac{-2\cos\theta + 2x}{1 - 2x\cos\theta + x^2}\, d\theta.$$

Clearly, $I'(0) = 0$. We now show that $I'(x) = 0$ for $x \neq 0$. First, we prove that

$$\int_0^{\pi} \frac{1 - x^2}{1 - 2x\cos\theta + x^2}\, d\theta = \pi. \qquad (4)$$

The integrand in (4) is called Poisson's kernel, which is used to derive solutions of the two-dimensional Laplace's equation on unit circle [**1**, p.135], and also plays an important role in summation of Fourier series [**2**, pp.163–170]. Computing the value of this integral is often used to show the usefulness of the residue theorem, a relatively advanced tool [**3**, p. 303]. We give a more straightforward method using the half-angle substitution. Setting $t = \tan(\theta/2)$, we have

$$\begin{aligned}\int \frac{1-x^2}{1-2x\cos\theta+x^2}\,d\theta &= 2(1-x^2)\int \frac{dt}{(1-x)^2+(1+x)^2t^2}\\ &= 2\arctan\left(\frac{1+x}{1-x}t\right)+C\\ &= 2\arctan\left(\frac{1+x}{1-x}\tan(\theta/2)\right)+C.\end{aligned}$$

The fundamental theorem of calculus then gives

$$\int_0^\pi \frac{1-x^2}{1-2x\cos\theta+x^2}\,d\theta = \lim_{\theta\to\pi} 2\arctan\left(\frac{1+x}{1-x}\tan(\theta/2)\right) = \pi.$$

Using equation (4) and $x \neq 0$, we get

$$I'(x) = \frac{1}{x}\int_0^\pi \left(1 - \frac{1-x^2}{1-2x\cos\theta+x^2}\right)d\theta = 0.$$

Thus, we have $I'(x) = 0$ for $|\,x\,| < 1$ and so $I(x) =$ constant. Since $I(0) = 0$, we have shown that $I(x) \equiv 0$, for $|\,x\,| < 1$.

IV. Using infinite series We first show that

$$\ln(1-2x\cos\theta+x^2) = -2\sum_{n=1}^{\infty}\frac{x^n}{n}\cos n\theta, \tag{5}$$

where the series converges uniformly for $|\,x\,| < 1$. Once we establish this, integrating (5) with respect to θ from 0 to π, will show that $I(x) = 0$ once again.

To prove (5) and to keep the evaluation at an elementary level, instead of using the Fourier series, we use the relation $2\cos\theta = e^{i\theta} + e^{-i\theta}$ and decompose into partial fractions:

$$\frac{1-x^2}{1-2x\cos\theta+x^2} = \frac{1-x^2}{(1-xe^{i\theta})(1-xe^{-i\theta})} = -1 + \frac{1}{1-xe^{i\theta}} + \frac{1}{1-xe^{-i\theta}}.$$

Then the geometric series expansion leads to,

$$\frac{1-x^2}{1-2x\cos\theta+x^2} = 1 + 2\sum_{n=1}^{\infty} x^n\cos n\theta. \tag{6}$$

The series (6) converges uniformly since $\sum_{n=1}^{\infty} |\,x\,|^n$ converges for $|\,x\,| < 1$. Subtracting 1 from the right-hand side of (6) and then dividing by x, we have

$$\frac{2\cos\theta-2x}{1-2x\cos\theta+x^2} = 2\sum_{n=1}^{\infty} x^{n-1}\cos n\theta. \tag{7}$$

Since the series (7) converges uniformly, integrating from 0 to x term by term, we have established (5) as desired.

Remark As a bonus, we have another proof of integral (4) deduced from series (6).

We have seen a variety of evaluations of the Poisson integral. The interested reader is encouraged to investigate additional approaches.

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Cycles in the Generalized Fibonacci Sequence modulo a Prime

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Since their invention in the thirteenth century, Fibonacci sequences have intrigued mathematicians. As well as modeling the population patterns of overly energetic rabbits, however, they have sparked developments in more serious mathematics. For example, generalized Fibonacci sequences crop up in all manner of situations, from fiber optic networks [**3**] to computer algorithms [**1**] to probability theory [**2**].

In this article, we study generalized Fibonacci sequences $\{G(n)\}$, given by the recurrence relation: $G(n) = aG(n-1) + bG(n-2)$ for a, b, $G(0)$ and $G(1)$ integers. We also study the periods of repetition in such sequences when considered modulo p, a prime. For one particular class of generalized Fibonacci numbers, we find a surprising connection with Fermat's Last Theorem. Other connections between these two seemingly unrelated subjects have been discovered in the past [**8**], but the one unearthed here allows us to calculate the length of these repetitions or *cycles* exactly.

Some useful results When working with the generalized Fibonacci sequences described above, we will need some results to make our lives easier. It is well known that the usual Fibonacci numbers (that is $a = b = 1$, $G(0) = 0$, $G(1) = 1$) can be expressed using Binet's formula [**5**]:

$$\sqrt{5}G(n) = \left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

This is easily derived by guessing a solution to the recurrence of the form $G(n) = \lambda^n$, solving for λ, and matching with the initial conditions. In a similar way, many number theory texts (see for example, Niven and Zuckermann [**7**]) prove that the analogous Binet formula for our sequence is

$$(A-B)G(n) = G(1)(A^n - B^n) + G(0)(AB^n - BA^n), \tag{1}$$

where

$$A = \frac{a + \sqrt{a^2 + 4b}}{2}, B = \frac{a - \sqrt{a^2 + 4b}}{2},$$

provided $a^2 + 4b \neq 0$.

In this article, we will focus on those generalized Fibonacci sequences where A and B are both integers. We do this because it will lead us to nicer and more exact results later, but first we should investigate what it means for our sequence. If $a^2 + 4b = x^2$ (where x is an integer) then x has the same parity as a. The fact that A and B are both integers follows immediately from this. Conversely, since $a^2 + 4b = (2A - a)^2$, then if A and B are both integers it is clear that $a^2 + 4b$ is a perfect square. This is the first of our useful results and we summarize it as:

(I) A and B are integers if and only if $a^2 + 4b \neq 0$ is a perfect square.

In order to obtain more results concisely, we can also use Fibonacci matrices that are defined to be

$$\mathbf{G_k} = \begin{pmatrix} G(k+2) & G(k+1) \\ G(k+1) & G(k) \end{pmatrix}, \ \mathbf{M} = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}.$$

A simple induction shows that $\mathbf{G}_n = \mathbf{M}^n \mathbf{G}_0$. If we then take determinants of this equation we have the second of our results:

(II) $G(n+2)G(n) - G(n+1)^2 = (-b)^n (G(2)G(0) - G(1)^2)$.

Now that we have these results at our disposal, we can begin to answer the real question we are interested in: What can we say about these generalized Fibonacci sequences when looking at them modulo p?

Finding points of repetition in the sequence Using (1) in conjunction with the result (I), we are able to discover some interesting divisibility properties of this class of Fibonacci sequence. Central to much of this work is Fermat's Little Theorem, which states that $x^p \equiv x (\text{mod} p)$ for all primes p and integers x. Because this result only works for integers, we need to ensure that A and B are both integers, that is, that $a^2 + 4b = x^2$, for some nonzero integer x, and we shall continue to do so for the rest of the article. Reducing the Binet analog (Equation 1) modulo p, and using Fermat's Little Theorem, it is easy to show that $G(p) \equiv G(1)(\text{mod} p)$ if $(x, p) = 1$. A similar argument, involving such easy formulas as $A^2 - B^2 = ax$, shows that

$$G(p+1) \equiv (aG(1) + bG(0)) \, (\text{mod} p) \equiv G(2)(\text{mod} p).$$

But, $G(p+1) = aG(p) + bG(p-1)$, so that

$$bG(p-1) \equiv G(2) - aG(1)(\text{mod} p) \equiv bG(0).$$

If $(b, p) = 1$, we can conclude that $G(p-1) \equiv G(0)(\text{mod} p)$. The conditions $(x, p) = (b, p) = 1$ are so important in the work we do that if p is a prime satisfying them, we say that p is a *Level I prime with respect to the sequence generated by a and b* abbreviated to "p is LI wrt a, b." The last result can now be written:

(III) If p is LI wrt a, b then $G(p-1) \equiv G(0)(\text{mod} p)$ and $G(p) \equiv G(1)(\text{mod} p)$.

What about cycles then? Before we begin to think about cycles, it is important to have clear in our minds what we mean by the cycle length of the generalized Fibonacci sequence. In general, sequences that eventually settle down into nice cycles can exhibit strange behavior for a finite number of terms. So we define the set $\mathcal{C}_p$ as follows, in order to identify *eventual* cycles in the sequence:

$$\mathcal{C}_p = \{c > 0 \mid G(n+c) \equiv G(n), G(n+c+1) \equiv G(n+1), \text{for } n \text{ sufficiently large}\}$$

Here, as in the rest of the paper, all equivalences are to be understood as being mod p. Then we set $C(p) = \min \mathcal{C}_p$; this is what we mean by the cycle length of a particular sequence repeating modulo p. (The dependence of $C(p)$ on a and b is implicit.)

The most interesting results are obtained by considering only the Level I primes with respect to the sequences generated by a and b. Perhaps the most obvious questions to ask about the sequences reduced modulo these primes are (i) Did we really need to worry about cycles beginning far out on the sequence? and (ii) Is there a finite cycle length at all?

The answer is affirmative to each of these questions and, at least for question (ii), it is not too hard to see why. There are only a total of p^2 different pairings of numbers in the sequence (since we are dealing only with the numbers $0, \ldots, (p-1)$) and so we must have $C(p) \leq p^2$.

Question (i) is a little harder to answer, so we shall do this more slowly! Suppose that the cyclic behavior begins with the nth term, so that

$$G(n+c) \equiv G(n), \text{ and } G(n+c+1) \equiv G(n+1),$$

but $G(n-1+c) \not\equiv G(n-1)$. Expressing $G(n+c-1)$ and $G(n-1)$ in terms of their successors, using the recursion relation, gives $bG(n+c-1) \equiv bG(n-1)$. As $(b, p) = 1$, we see that the cycle must begin at the beginning, with $G(C(p)) \equiv G(0)$, $G(C(p)+1) \equiv G(1)$, as we would have liked.

p^2 can't be the best bound! There is indeed a better bound on the cycle length than p^2 (We already knew this for Level I primes, since by (3) a cycle must have occurred by $p-1$). In fact, we can do substantially better than that and in some cases, we can even calculate the exact cycle length, but that will have to wait for a few more sections. In the meantime, we shall obtain results analogous to those obtained by many other people for the normal and generalized Fibonacci sequences. The seminal articles on this subject are those by Wall **[9]** (who invented the subject!) and Li **[6]**, although the results we obtain are in some ways nicer.

To demonstrate this kind of sequence, we look at the case $a = 1, b = 2$ with $G(0) = 0$, $G(1) = 1$. This gives us the generalized Fibonacci sequence beginning

$$0, 1, 1, 3, 5, 11, 21, 43, \ldots.$$

Reduced modulo 11, which is LI wrt 1, 2, we have the sequence $0, 1, 1, 3, 5, 0, -1, -1, -3, -5, 0, 1, 1, 3, 5, \ldots$.

This sequence clearly repeats every tenth term, so we write $C(11) = 10$. The result we now obtain will help to explain why the answer here is 10, though we will not know the full story until later. By (3), we know that $C(p) \leq p-1$, and we can write

$$G(kC(p)) \equiv G(p-1) \quad \text{for all integers } k \geq 0.$$

If we assume that $C(p)$ is not a divisor of $p-1$ and let j be the floor of $(p-1)/C(p)$, that is,

$$j = \left\lfloor \frac{p-1}{C(p)} \right\rfloor \neq \frac{p-1}{C(p)},$$

then we find that we have a repetition in the Fibonacci sequence between $G(jC(p))$ and $G(p-1)$ that is shorter than $C(p)$. This contradicts the fact that $C(p)$ was taken to be the smallest cycle length, and so we conclude that $C(p)|p-1$. This gives us one of our most important results:

(IV) For all LI primes p wrt a, b, $C(p) \mid p-1$.

The theorem corresponding to this last result in Wall's 1960 paper **[9]** on the normal Fibonacci numbers has a more complicated result than ours, as we had the luxury of looking at the case where A and B are integers. The reason we say our results are nicer is that $a^2 + 4b = x^2$ and so is a quadratic residue of all primes whereas 5 (which is $a^2 + 4b$ for the usual Fibonacci numbers) is only a quadratic residue of primes of the form $10n \pm 1$.

In the example at the beginning of this section we had $a = 1$, $b = 2$, $G(0) = 0$, $G(1) = 1$, and $C(11) = 10$. In this case, our upper bound for $C(p)$ is met but for the time being we are unable to provide a similar lower bound. As we shall see, however, it is no accident that in this particular case the upper bound is met.

Another viewpoint on cycles If we want to progress much farther, it is useful to try and understand what causes a Fibonacci cycle. This is a very deep question and we can only partially answer it here. Primarily, we are interested in the order of A and B modulo p ($\text{ord}_p(A)$, etc.), where the *order* is the smallest nontrivial exponent e such that, for example, $A^e \equiv 1 (\text{mod}\, p)$. If $\text{ord}_p(A) \mid r$ and $\text{ord}_p(B) \mid r$, then the Binet analog shows that terms r and $(r+1)$ of our sequence are

$$G(r) \equiv \frac{1}{A-B}\left(G(1)\,(1-1) + G(0)\,(A-B)\right) \equiv G(0)$$

$$G(r+1) \equiv \frac{1}{A-B}\left(G(1)\,(A-B) + G(0)\,(AB-BA)\right) \equiv G(1)$$

so that a cycle begins at r.

Conversely, if $G(r) \equiv G(0)$, $G(r+1) \equiv G(1)$, then we have:

$$\begin{aligned} G(0)(A-B) &\equiv G(1)(A^r - B^r) + G(0)(AB^r - BA^r) \\ G(1)(A-B) &\equiv G(1)(A^{r+1} - B^{r+1}) + G(0)(AB^{r+1} - BA^{r+1}) \\ &\Rightarrow (A-B)(G(1) - AG(0)) \equiv G(1)(B^r A - B^{r+1}) \\ &\quad + G(0)(AB^{r+1} - B^r A^2) \\ &\Rightarrow G(1)(A-B)(B^r - 1) + AG(0)(A-B)(1-B^r) \equiv 0 \\ &\Rightarrow (G(1) - AG(0))(A-B)(B^r - 1) \equiv 0 \end{aligned}$$

Thus $B^r \equiv 1$, that is, $\text{ord}_p(B) \mid r$, or $G(1) \equiv AG(0)$. By the symmetry of the equations we also have: $A^r \equiv 1$, that is, $\text{ord}_p(A) \mid r$, or $G(1) \equiv BG(0)$. However, we can rule out these possibilities by insisting that our sequence is such that $(G(1) - AG(0), p) = 1$ and similarly for the second condition. Level I primes that satisfy these conditions as well are known as *Level II primes wrt the sequence generated by* a, b, $G(0)$ *and* $G(1)$. We will abbreviate this in a way similar to that for LI primes: we say that p is LII wrt $a, b, G(0)$ and $G(1)$. Under this new requirement, we can also deduce easily that $C(p) = [\text{ord}_p(A), \text{ord}_p(B)]$ where $[x, y]$ denotes the lowest common multiple of x and y, giving us result (5):

(V) For p, LII wrt $a, b, G(0)$ and $G(1)$, a cycle begins at r iff $\text{ord}_p(A) \mid r$ and $\text{ord}_p(B) \mid r$. Furthermore, $C(p) = [\text{ord}_p(A), \text{ord}_p(B)]$.

This result reduces to the question of cycle lengths to another well-established one in number theory. As yet, there is no answer to the seemingly simple question of determining $\text{ord}_p(n)$, although we can answer the question of what the cycle length is in some cases, as we shall see in the last section. First, however, it is interesting to note that this result implies that the cycle length for these kinds of sequence is independent of the starting points $G(0)$ and $G(1)$ (as long as the conditions are met of course!) since it depends only on the orders of A and B.

A strange connection with Fermat's Last Theorem?! So far we have been unable to calculate the exact cycle length of a generalized Fibonacci sequence modulo any prime. However, for many values of A and B (and hence a and b) it is possible to prove that the order of at least one of A and B will be even for any prime. Perhaps the most obvious example of this is when one of A or B is -1. It is interesting to see what the condition $A = -1$ or $B = -1$ really means for the kind of Fibonacci recurrence relation we are interested in. This is a simple matter as we may take (without loss of generality) $B = -1$. The numbers a and b are given in terms of A as follows:

$$-b = AB = -A \text{ and } a = A + B = A - 1 = b - 1,$$

so that all recurrence relations of the form

$$G(n) = aG(n-1) + (a+1)G(n-2) \tag{2}$$

have one of A or $B = -1$. By result (IV) in the last section, this in turn means that the cycle lengths for these types of sequence are always even (for any prime p). This result, innocuous as it may seem, will now enable us to calculate the exact cycle length of this type of sequence for certain primes. This will require a new definition and will reveal a slightly surprising link with Fermat's Last Theorem!

DEFINITION. If p, a prime, is such that $p = 2q + 1$ where q is also a prime, then q is a *Sophie Germain prime* of the first kind and p is a reverse Sophie Germain prime. Sophie Germain used her primes (see Wells [**10**] for a list) to prove a particular case of Fermat's Last Theorem, a strategy that was later developed by Lagrange to include more cases. Since Wiles' proof of this theorem, however, the study of these primes is now little more than a mathematical curiosity. However, it is the concept of reverse Sophie Germain primes that interests us here. If we are looking at a generalized Fibonacci sequence (of the type described above) modulo p, where p is a reverse Sophie Germain prime, then since $C(p) \mid p - 1 (= 2q)$ and $C(p)$ must be even, the only options for the cycle length are $C(p) = 2$, $C(p) = p - 1$. If we can eliminate the first of these possibilities then we will know the exact cycle length for a whole host of primes! The shortest way to do this is simply to state as a condition that $C(p) \neq 2$, although it is not too difficult (using Cramer's rule) to show that an equivalent set of conditions on a, b, $G(0)$ and $G(1)$ is

$$a + b \not\equiv 1, \text{ if } G(0) \equiv G(1)$$
$$b - a \not\equiv 1 \text{ if } G(0) \equiv -G(1)$$
$$\text{and } a \not\equiv 0 \text{ or } b \not\equiv 1 \text{ if } G(0) \not\equiv \pm G(1).$$

With this condition we have the surprising result:

(VI) If we have a generalized Fibonacci sequence given by (2), then for reverse Sophie Germain primes, $C(p) = p - 1$, subject to the condition that $C(p) \neq 2$ and that p is also LII wrt a, b, $G(0)$ and $G(1)$.

It is now clear that the particular sequence we looked at earlier (with $a = 1, b = 2$, $G(0) = 0$, $G(1) = 1$ and $p = 11$) is one of these sequences and so we could have predicted that $C(11) = 10$ after only having checked that $C(11) \neq 2$ (we could do this by calculating the third and fourth terms of the sequence).

In the interests of being able to apply this result to a whole host of primes for a particular sequence we have neglected the fact that we can also have, in exactly the same way:

(VII) If $A \equiv -1 (\text{mod} p)$ or $B \equiv -1 (\text{mod} p)$, then $C(p)$ is even. Further, if p is a reverse Sophie Germain prime then $C(p) = p - 1$, provided that $C(p) \neq 2$ and that p is LII wrt a, b, $G(0)$ and $G(1)$.

It is believed that there may be infinitely many Sophie Germain primes (and hence reverse Sophie Germain primes!). The question of whether this is true is related to the conjecture that there are infinitely many twin primes and, like it, does not yet have a solution. However, a great deal of work has been carried out on the calculation of large Sophie Germain primes [**4**]. Because of this a very large number of such primes are known and hence we can calculate the exact cycle length of a large number of primes for certain generalized Fibonacci sequences.

Although we have only been interested in the case where $a^2 + 4b$ is a perfect square, it is possible to replace this by looking only at the prime residue systems in which $a^2 + 4b$ is a quadratic residue. The results we have obtained all follow through in this case, it is just more complicated to state them and give their proofs although the keen reader will have little trouble in verifying this.

Note One interesting reference not referred to in the text is Neil Sloane's homepage (http://www.research.att.com/~njas) which has links to his encyclopedia of number sequences. Here there are many examples of Fibonacci (and Lucas) sequences as well as some examples of Sophie Germain primes and related sequences called Cunningham chains (the Cunningham chain with the smallest elements is $2, 5 = 2 \times 2 + 1$, $11 = 2 \times 5 + 1, 23 = 2 \times 11 + 1, 47 = 2 \times 23 + 1$).

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Products of Chord Lengths of an Ellipse

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Suppose we choose $n > 1$ equally spaced points on the unit circle, dividing it into n equal arcs. Choose one of the points as a base point and draw chords connecting it to each of the other points. (FIGURE 1 shows the case $n = 8$.) Then the product of the lengths of these chords equals n.

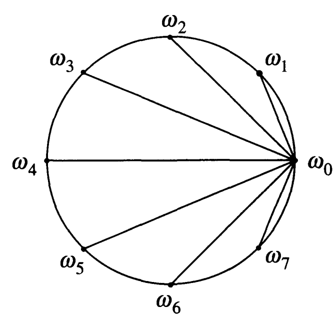

Figure 1 Unit circle with seven chords

This often-rediscovered result first appeared, with a trigonometric proof, in a paper by Sichardt [3] about hydraulics. Mazzoleni and Shen [2] gave a proof using the residue theorem. Actually, the product is easily determined using polynomials as follows. Construct the chords using the nth roots of unity ω^j, $j = 0, \dots, n-1$ with base point $\omega_0 = 1$. Since these points are the roots of the polynomial $z^n - 1$, the desired product is then given by the absolute value of

$$\prod_{j=1}^{n-1} \left(\omega_0 - \omega_j\right) = \lim_{z \to 1} \frac{z^n - 1}{z - 1} = \frac{d}{dz}(z^n - 1)\bigg|_{z=1} = n. \tag{1}$$

Thus, the result follows from calculating the derivative of a polynomial with carefully chosen roots. In this note, we employ a more general triangular family of polynomials and generalize Sichardt's result to chords of an ellipse.

Elliptical preliminaries Let $a \geq b \geq 0$ and, as usual, let the complex number $z = u + iv$ (where $i := \sqrt{-1}$) represent the ordered pair (u, v). The locus of points

$$ae^{i\theta} + be^{-i\theta} = (a+b)\cos\theta + i(a-b)\sin\theta, \tag{2}$$

$(0 \leq \theta < 2\pi)$ describes an ellipse with major axis vertices $\pm(a+b)$ and minor axis vertices $\pm i(a-b)$. For appropriate choices of a and b, equation (2) describes any ellipse with foci equidistant from the origin on the real axis.

To construct chords, initially we choose the n points on the ellipse that are the images of the nth roots of unity under the mapping

$$e^{i\theta} \longrightarrow ae^{i\theta} + be^{-i\theta} \tag{3}$$

with real base point $z_0 := a + b$, the right-most vertex of the ellipse. (FIGURE 2 shows the case $n = 8$.) Unless $b = 0$, which yields the circular case already studied, the elliptical arcs determined by these points *do not* have equal arc length.

An immediate consequence of our main result, established in what follows, is that the product d_n of the lengths of the resulting chords is given by

$$d_n = n\frac{a^n - b^n}{a - b}. \tag{4}$$

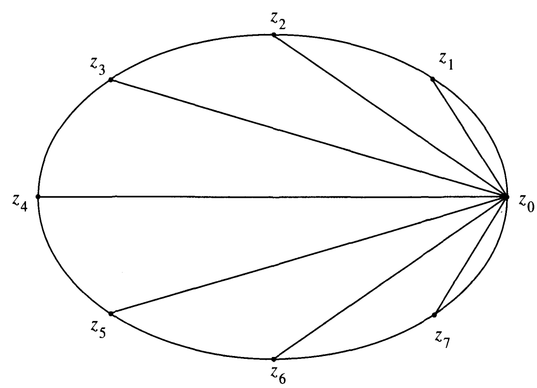

Figure 2 Ellipse with seven chords

The dependence of this value (and similar formulas that follow) on a and b should be clear from the context. The following example demonstrates formula (4) when applied to the special case of a circle.

EXAMPLE 1. *Suppose $b = 0$, so that the ellipse reduces to a circle with radius a. Setting $z_0 = a$ we have by equation (4) that $d_n = na^{n-1}$; this yields equation (1) when $a = 1$. Also observe that*

$$d_n = \frac{d}{dz}(z^n - a^n)\bigg|_{z=a}.$$

Let us standardize the notation $\theta_j := 2j\pi/n$ and $z_j := ae^{i\theta_j} + be^{-i\theta_j}$ for $j = 0, \ldots, n-1$ throughout this paper. Note that θ_j, $j = 0, \ldots, n-1$, are the arguments of the nth roots of unity so that the set $\{z_j\}$ is the image of these roots under the mapping (3). The product of the lengths of the $n-1$ line segments from any point z in the complex plane to z_j, $j \neq 0$, is denoted by

$$d_n(z) = \prod_{j=1}^{n-1} | z - z_j |. \tag{5}$$

In particular, $d_n = d_n(z_0)$ where d_n is the value given in equation (4).

Some useful polynomials The products defined by Equation (5) might suggest that we study the polynomials $\prod_{j=0}^{n-1}(z - z_j)$, $n = 1, 2, \ldots$. These polynomials can be generated recursively, but we offer a more attractive algorithm for computing a slightly different family:

$$P_n(z) := \prod_{j=0}^{n-1}(z - z_j) + (a^n + b^n), \quad n = 1, 2, \ldots. \tag{6}$$

The introduction of the term $a^n + b^n$ causes no additional complications, since for any complex number z,

$$d_n(z) = \begin{cases} \left|\dfrac{P_n(z) - (a^n + b^n)}{z - z_0}\right| & \text{if} \quad z \neq z_0 \\ |P_n'(z_0)| = \lim\limits_{z \to z_0}\left|\dfrac{P_n(z) - (a^n + b^n)}{z - z_0}\right| & \text{if} \quad z = z_0 \end{cases}. \tag{7}$$

Notice that

$$P_n(z_j) = a^n + b^n, \quad j = 0, \ldots, n-1. \tag{8}$$

This characterizes $P_n(z)$ as the *unique* monic polynomial interpolant of degree $n \in \mathbb{N}$ to the constant $(a^n + b^n)$ in the nodes z_j, $j = 0, \ldots, n-1$. We will use this fact in determining a recurrence relation that will enable us to construct these polynomials explicitly.

Clearly, $P_1(z) = z$ satisfies (8) since $P_1(a + b) = a + b$. We will establish that the higher order polynomials satisfy the three-term recurrence relation

$$P_n(z) = zP_{n-1}(z) - abP_{n-2}(z), \quad n = 2, 3, \ldots, \tag{9}$$

where, for convenience, we set $P_0(x) := a^0 + b^0 = 2$. Since $\theta_j = 2j\pi/n$, so that $e^{\pm in\theta_j} = 1$ for $j = 0, \ldots, n-1$, equation (8) will follow if the polynomials determined by (9) satisfy

$$P_n(ae^{i\theta} + be^{-i\theta}) = a^n e^{in\theta} + b^n e^{-in\theta}, \quad 0 \leq \theta \leq 2\pi \ (n \geq 2). \tag{10}$$

Direct substitution establishes that (10) holds for $P_0(z)$ and $P_1(z)$. Next, assume that (10) is valid for all polynomials generated by (9) of degree less than $n \geq 2$. Then by induction equation (9) yields

$$\begin{aligned} P_n(ae^{i\theta} + be^{-i\theta}) &= (ae^{i\theta} + be^{-i\theta})(a^{n-1}e^{i(n-1)\theta} + b^{n-1}e^{-i(n-1)\theta}) \\ &\quad - ab(a^{n-2}e^{i(n-2)\theta} + b^{n-2}e^{-i(n-2)\theta}) \\ &= a^n e^{in\theta} + b^n e^{-in\theta}, \end{aligned}$$

establishing equation (10). The first four polynomials determined by equation (9) are

$$\begin{aligned} P_2(z) &= z^2 - 2ab \\ P_3(z) &= z^3 - 3abz \\ P_4(z) &= z^4 - 4abz^2 + 2a^2b^2 \\ P_5(z) &= z^5 - 5abz^3 + 5a^2b^2z. \end{aligned}$$

There are two facts worth noting at this point. First, for the special case $a = b = 1/2$, the polynomials $2^{n-1}P_n(z)$ are the well-known Tchebycheff polynomials of the first kind. Davis **[1]** gives a brief introduction to this class of polynomials. Second, the special case of equation (10) when $a = 1$ and $b = 0$, so that $P_n(z) = z^n$, is DeMoivre's theorem.

Determining the product of the chord lengths The limit in the second part of equation (7) is independent of the path on which z approaches z_0 so we may restrict z to the ellipse. In view of equation (10) we obtain

$$d_n(z_0) = \lim_{\theta\to 0}\left|\frac{(a^n e^{in\theta} + b^n e^{-in\theta}) - (a^n + b^n)}{(ae^{i\theta} + be^{-i\theta}) - (a+b)}\right|. \tag{11}$$

Applying l'Hôpital's Rule to the indeterminate limit (11) establishes equation (4).

The next example is an application of (7) taking the base point to be the top vertex, $z = i\,(a-b)$, of the ellipse. (FIGURE 3 shows the case $n = 3$. The dashed line segment in that figure indicates the chord whose length is removed from the calculation of $d_n[i\,(a-b)]$.)

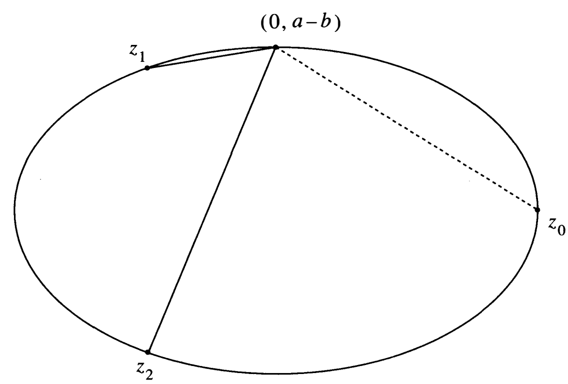

Figure 3 Drawing for Example 2

EXAMPLE 2. *Using $\theta = \pi/2$ in equation (10) gives*

$$P_n[i\,(a-b)] = i^n a^n + i^{-n} b^n.$$

A few calculations using the first part of equation (7) with $z = i\,(a-b)$ establish

$$d_n[i\,(a-b)] = \begin{cases} 0, & \text{if } n = 0 \bmod 4 \\ \sqrt{2}\,(a^n + b^n)/\sqrt{a^2+b^2}, & \text{if } n = 2 \bmod 4 \\ \sqrt{a^{2n}+b^{2n}}/\sqrt{a^2+b^2}, & \text{if } n = 1, 3 \bmod 4. \end{cases}$$

It is reasonable to compare the product of chord lengths of the unit circle with that of an ellipse provided that the ellipse is in some sense normalized. One possibility is to ensure that the ellipse has area π, the area of the unit circle. Since the area of the ellipse determined by $ae^{i\theta} + be^{-i\theta}, 0 \le \theta < 2\pi$, is $\pi(a+b)(a-b)$, it suffices to require that $(a+b)(a-b) = (a^2 - b^2) = 1$. Then $a = \sqrt{1+b^2} > 1$ unless $b = 0$, so for $b \neq 0$ we have from equation (4)

$$d_n(a+b) = n\sum_{j=0}^{n-1} a^{n-j-1}b^j = na^{n-1} + n\sum_{j=1}^{n-1} a^{n-j-1}b^j > na^{n-1} > d_n(1).$$

This says that the product of chord lengths for the unit circle is the infimum of the product of chord lengths for all comparable ellipses. This infimum is obtained when the conic is a circle of radius one. On the other hand, if a increases without bound,

$d_n(a+b) > na^{n-1}$ tends to infinity, so the product of chord lengths of ellipses with area π can be made as large as desired.

Rotating the endpoints of the chords FIGURE 2 suggests that unlike the case for the circle, the value of $d_n(z)$ for z on the ellipse *does change* with rotations of the endpoints of the chords. Fix $\psi \in [0, 2\pi)$ and rotate the endpoints of line segments on the ellipse using the transformation that maps $z_j = ae^{i\theta_j} + be^{-i\theta_j}$ to $z_j^\psi := ae^{i(\theta_j+\psi)} + be^{-i(\theta_j+\psi)}$, $j = 0, \ldots, n-1$. Observe that this transformation does *not* rotate each point on the ellipse by the angle ψ. In view of equation (10),

$$P_n(z_j^\psi) = a^n e^{in\psi} + b^n e^{-in\psi},\ j = 0, 1, \ldots, n-1.$$

Proceeding as before it is evident that the product of the $n-1$ line segments from a point z in the plane to the nodes z_j^ψ, $j = 1, \ldots, n-1$ is given by

$$d_n^\psi(z) := \begin{cases} \left| \dfrac{P_n(z) - (a^n e^{in\psi} + b^n e^{-in\psi})}{z - (ae^{i\psi} + be^{-i\psi})} \right| & \text{if} \quad z \neq z_0^\psi \\ | P_n'(z_0^\psi) | & \text{if} \quad z = z_0^\psi. \end{cases} \tag{12}$$

Once again, restricting z to the ellipse and passing to the limit $(\theta \to \psi)$ yields

$$d_n^\psi(z_0^\psi) = n \left(\frac{a^{2n} + b^{2n} - 2a^n b^n \cos(2n\psi)}{a^2 + b^2 - 2ab\cos(2\psi)} \right)^{\frac{1}{2}}. \tag{13}$$

By symmetry we can restrict ψ to $[0, \pi]$. Since $\cos(2(\pi/2 + \beta)) = \cos(2(\pi/2 - \beta))$, we can further restrict ψ to $[0, \pi/2]$. Note that if $\psi = \theta_k$ for some k, then (13) gives the value of the product of chord lengths using the points z_j, $j \neq k$, with base point $z_0^\psi = z_k$.

EXAMPLE 3. *If $\psi = \pi/2$ so that $z_0^{\pi/2} = i\,(a-b)$, equation (13) gives*

$$d_n^{\pi/2}[i\,(a-b)] = \begin{cases} n\dfrac{a^n + b^n}{a+b}, & \textit{if} \quad n \textit{ is odd} \\ n\dfrac{a^n - b^n}{a+b}, & \textit{if} \quad n \textit{ is even.} \end{cases}$$

Notice that these values are different from those obtained in Example 2 where the endpoints of the chords were not rotated.

EXAMPLE 4. *Some calculations require special handling if $z = z_j^\psi$ for some j. Suppose n is odd so that $z_j^0 \neq -(a+b)$ for any j. Then from equation (12)*

$$d_n^0[-(a+b)] = \frac{a^n + b^n}{a+b}.$$

However, if n is even, so that $z_{n/2}^0 = -(a+b)$, then $d_n^0[-(a+b)] = 0$ because the trivial line segment of length zero from the base point $-(a+b)$ to $z_{n/2}^0$ is included in the product. In the spirit of the original problem ([3]) it seems appropriate to include only nontrivial chords (those with positive length) in the calculation. This can be handled by exploiting the symmetry of the points z_j^0 for even values of $n \geq 2$ and computing

$$d_n^0[-(a+b)] = \frac{d_n(a+b)}{2(a+b)} = \frac{n(a^n-b^n)}{2(a^2-b^2)}.$$

The factor $2(a+b)$ *accounts for the length of the line segment connecting* z_0^0 *to* $z_{n/2}^0$. *This factor is included in the product* $d_n(a+b)$ *but should not be a part of the calculation of* $d_n^0[-(a+b)]$.

Maximum value of d_n^ψ for z on the ellipse Recall that for $b = 0$ the ellipse reduces to a circle and $P_n(z) = z^n$ for all positive integers n. Then

$$d_n(z) = \left|\frac{z^n - a^n}{z-a}\right| = \left|\sum_{j=0}^{n-1} z^{n-j-1}a^j\right| \le \sum_{j=0}^{n-1} a^{n-1} = na^{n-1} = d_n(a) \tag{14}$$

indicating that $d_n(a) = na^{n-1}$ is the maximum value of $d_n(z)$ for all complex numbers z on the circle of radius a. Surprisingly, a similar result holds for the ellipse even when rotations of the type described in the previous section are allowed. Specifically, as will be proven below,

$$d_n^\psi(z) = \left|\frac{P_n(z) - P_n(w)}{z-w}\right| \le P_n'(a+b) = n\frac{a^n-b^n}{a-b} = d_n(a+b), \tag{15}$$

where $w = (ae^{i\psi} + be^{-i\psi})$ and $z = ae^{i\theta} + be^{-i\theta}$ are any points on the ellipse.

The argument establishing (15) is sufficiently complicated to justify a proof. There is also an annoying factor of $1/2$ that arises somewhat unnaturally in the calculations. In what follows, a prime ($'$) on summation symbols indicates that any term in the sum with index $j = 0$ or $j = n-1$ is to be multiplied by $1/2$.

We will need the identity

$$\frac{P_n(z) - P_n(w)}{(z-w)} = {\sum_{j=0}^{n-1}}' P_{n-j-1}(z)P_j(w) + \frac{ab[P_{n-2}(z) - P_{n-2}(w)]}{(z-w)}. \tag{16}$$

The first step in proving equation (16) is to verify the equality

$$z\sum_{j=1}^{n-2} P_{n-j-1}(z)P_j(w) = \sum_{j=1}^{n-2} P_{n-j}(z)P_j(w) + ab\sum_{j=1}^{n-2} P_{n-j-2}(z)P_j(w). \tag{17}$$

This can be done by first solving for $zP_{n-j-1}(z)$, $j = 1, \ldots, n-2$, in the recurrence relation (9) with appropriate changes in the subscripts. Next, multiply the resulting equation by $P_j(w)$ and then sum over j. (Recall that $P_0(z) \equiv 2$.) Interchanging z and w in equation (17) and reordering the resulting sums verify that

$$w\sum_{j=1}^{n-2} P_{n-j-1}(z)P_j(w) = \sum_{j=2}^{n-1} P_{n-j}(z)P_j(w) + ab\sum_{j=0}^{n-3} P_{n-j-2}(z)P_j(w). \tag{18}$$

Subtracting (18) from (17) produces the equation

$$\begin{aligned} S(z,w) :&= (z-w)\sum_{j=1}^{n-2} P_{n-j-1}(z)P_j(w) \\ &= wP_{n-1}(z) - zP_{n-1}(w) + 2ab[P_{n-2}(w) - P_{n-2}(z)]. \end{aligned}$$

This last identity and the recurrence relation (9) suggest

$$(z-w)\sum_{j=0}^{n-1}{}' P_{n-j-1}(z)P_j(w) = (z-w)\,[P_{n-1}(z)+P_{n-1}(w)] + S(z,w)$$

$$= zP_{n-1}(z) - abP_{n-2}(z) - [wP_{n-1}(w) - abP_{n-2}(w)]$$

$$+ ab[P_{n-2}(w) - P_{n-2}(z)]$$

$$= P_n(z) - P_n(w) + ab\,[P_{n-2}(z) - P_{n-2}(w)],$$

which is equivalent to equation (16).

The final step in establishing equation (15) requires induction. This equation can be directly verified for $n = 2, 3$. To see how this is done for the case $n = 3$ recall equation (15) and observe that

$$\begin{aligned} d_3^{\psi}(z) &= \left|\frac{P_3(z) - P_3(w)}{z-w}\right| = \left|\frac{z^3 - 3abz - (w^3 - 3abw)}{z-w}\right| \\ &= |\,z^2 + zw + w^2 - 3ab\,| \\ &\le |\,z^2 - 2ab\,| + |\,w^2 - 2ab\,| + |\,zw + ab\,| \\ &= |\,a^2e^{i2\theta} + b^2e^{-i2\theta}\,| + |\,a^2e^{i2\psi} + b^2e^{-i2\psi}\,| + |\,zw + ab\,| \\ &\le 2(a^2+b^2) + (a+b)^2 + ab \\ &= 3(a^2+ab+b^2) = 3\frac{a^3-b^3}{a-b}. \end{aligned}$$

Now fix $n > 3$ and assume equation (15) holds for $k < n$. Recalling equation (10), note that by (12), (16), and the induction hypothesis

$$\begin{aligned} d_n^{\psi}(z) &= \left|\sum_{j=0}^{n-1}{}' P_{n-j-1}(z)P_j(w) + \frac{ab[P_{n-2}(z) - P_{n-2}(w)]}{(z-w)}\right| \\ &\le \sum_{j=0}^{n-1}{}' \left[\left|\,a^{n-j-1}e^{i(n-j-1)\theta} + b^{n-j-1}e^{-i(n-j-1)\theta}\,\right|\left|\,a^je^{ij\psi} + b^je^{-ij\psi}\,\right|\right] \\ &\quad + ab\,d_{n-2}^{\psi}(z) \\ &\le \sum_{j=0}^{n-1}{}' \left[\left(a^{n-j-1} + b^{n-j-1}\right)\left(a^j + b^j\right)\right] + ab(n-2)\frac{a^{n-2} - b^{n-2}}{a-b} \\ &= (n-2)(a^{n-1} + b^{n-1}) + 2\frac{a^n - b^n}{a-b} + (n-2)\frac{a^{n-1}b - ab^{n-1}}{a-b}. \end{aligned}$$

Finally, combining the terms of this last expression produces the right side of equation (15). That $d_n\,(a+b)$ is the maximum value of $d_n\,(z)$ for all z on the ellipse given by equation (2) follows.

The inequality

$$\left(\frac{a^{2n} + b^{2n} - 2a^nb^n\cos(2n\psi)}{a^2 + b^2 - 2ab\cos(2\psi)}\right)^{1/2} \le \frac{a^n - b^n}{a-b}$$

for any real number ψ is an interesting and immediate consequence of equations (4), (13), and (15). One referee who reviewed this note spent some time thinking about the appearance of the Law of Cosines in this last inequality and mentioned that it may be pointing toward a "nicer or deeper result." I have also given this matter some thought which, unfortunately, did not spark additional insight. Readers may wish to examine this problem further.

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Surprisingly Accurate Rational Approximations

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Decimal digit accuracy in approximations The history of the number π reveals two rational approximations that are striking for their simplicity and accuracy: Archimedes' estimate 22/7, which gives two-decimal accuracy, and the Chinese estimate 355/113, which gives six decimals. In each case, the number of correct decimals is twice the number of digits in the denominator of the approximating rational. This is not merely a coincidence, nor is it a property unique to π. This note uncovers the surprising fact that every irrational can be approximated very closely by a rational whose denominator has a number of digits very nearly equal to half the number of decimal digits secured by the approximation.

Before stating this result as a theorem, we note further examples illustrating the phenomenon. In each case, the decimal digits secured by the rational approximation are underlined.

$$e - 2 = 0.71828182846\ldots \text{ is approximated}$$

$$\text{with 2-digit accuracy by } \frac{5}{7} = 0.\underline{71}42857\ldots,$$

$$\text{with 3-digit accuracy by } \frac{23}{32} = 0.\underline{718}75,$$

$$\text{and with 9-digit accuracy by } \frac{12{,}993}{18{,}089} = 0.\underline{718281828}74\ldots.$$

$$\sqrt{2} - 1 = 0.41421356237\ldots \text{ is approximated}$$

$$\text{with 4-digit accuracy by } \frac{29}{70} = 0.\underline{4142}857\ldots,$$

$$\text{and with 7-digit accuracy by } \frac{2378}{5741} = 0.\underline{4142135}5165\ldots.$$

$\log 2 = 0.301029995664\ldots$ is approximated

$$\text{with 5-digit accuracy by } \frac{87}{289} = 0.\underline{3010}380\ldots,$$

$$\text{and with 10-digit accuracy by } \frac{21{,}306}{70{,}777} = 0.\underline{3010299956}20\ldots.$$

The concept of decimal digit accuracy is subject to different interpretations. For example, we can say that $\pi = 3.141592653\ldots$ is approximated to 2 digits by $22/7 = 3.142857\ldots$ because

$$\left|\pi - \frac{22}{7}\right| = 0.00126\cdots < \frac{1}{10^2}.$$

This follows from the fact that the first two digits (.14) after the decimal point are identical in the decimal versions of π and 22/7. Similarly, π is approximated to six digits by the rational approximation $355/113 = 3.14159292035\ldots$ because

$$\left|\pi - \frac{355}{113}\right| = 0.000000267\cdots < \frac{1}{10^6}.$$

In this case, the first six digits .141592 after the decimal point are identical. On the other hand, the two numbers 0.199991 and 0.200000 do not agree in any of their digits after the decimal point, yet they differ only by 0.00009, and it is reasonable to assert that they agree with 4-digit accuracy.

In this note we say that two real numbers α and r agree with K-digit accuracy if

$$|\alpha - r| < \frac{1}{10^K}.$$

By truncating the decimal representation of an irrational after K decimals we get a rational approximation p/q with K-digit accuracy. The rational has the form $p/10^K$, so the denominator of this rational will contain $K+1$ digits if the fraction is in lowest terms. We are particularly interested in rational approximations where the number of digits in the denominator is much smaller than might be expected.

For example, truncating the decimal expansion for π after 5 decimals gives the rational approximation 314,159/100,000 with 5-digit accuracy. But a much better approximation (with 6-digit accuracy) is given by the rational 355/113 whose denominator has only 3 digits.

More examples of this phenomenon are mentioned above.

Approximating with continued fractions Continued fractions give excellent rational approximations to irrational numbers. In the previous issue of the MAGAZINE, Naylor [**1**] showed how they arise in the study of certain spirals in nature. We obtain these approximations in the following manner.

Every real number x can be written as a sum, $x = [x] + \{x\}$, where $[x]$ is the largest integer $\leq x$, and $\{x\}$ denotes the difference $x - \{x\}$, called the fractional part of x. If x is an integer the fractional part $\{x\}$ is zero. If x is not an integer, the fractional part is positive and less than 1, so it can be written as $1/y$ for some real $y > 1$. The number y in turn can be written as the sum $[y] + \{y\}$ giving

$$x = [x] + \frac{1}{y} = [x] + \frac{1}{[y] + \{y\}}.$$

If $\{y\} = 0$, then $x = [x] + 1/[y]$, a rational number, and the continued fraction terminates. But if $\{y\} > 0$, then $\{y\} = 1/z$ for some $z > 1$, and

$$x = [x] + \frac{1}{[y] + \frac{1}{[z]+\{z\}}}.$$

Repeating this process gives a continued fraction representation of x:

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}},$$

where $a_0 = [x]$, $a_1 = [y]$, $a_2 = [z]$, and so on. For ease in printing, this is written more compactly as $x = [a_0; a_1, a_2, a_3, \ldots]$, where all the integers after a_0 are positive. If x is rational the process terminates after a finite number of steps and the continued fraction is finite. But if x is irrational the continued fraction is infinite. For example, it is known that the continued fraction for π begins as follows:

$$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, \ldots].$$

Truncating the continued fraction of an irrational number α at the integer a_n produces a rational number

$$\frac{p_n}{q_n} = [a_0; a_1, \ldots, a_n]$$

called the nth convergent to α. Continued fractions are useful because p_n/q_n is closer to α than any other fraction p/q with denominator not exceeding q_n. For example, the third convergent to π is

$$[3; 7, 15, 1] = \frac{355}{113},$$

a fraction closer to π than any other fraction with denominator less than 113. Moreover, it can be shown [**2**, p. 335–6] that the nth convergent p_n/q_n to any irrational α is smaller than α if n is even, larger than α if n is odd, and no farther away from α than $1/(q_n q_{n+1})$, as summarized by the inequality

$$\left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_n q_{n+1}}. \tag{1}$$

From this inequality we can easily deduce the following theorem, which interprets this inequality in base ten. The proof is very simple and provides a practical application of inequality (1). The authors could not find this result in any of the literature on continued fractions, so it may be new.

THEOREM. *Let p_n/q_n be the nth convergent of a continued fraction for an irrational number α, and let K be the number of digits in the denominator q_n. Then the decimal representations of p_n/q_n and α agree with at least*

(a) $(2K - 2)$-digit accuracy for every n,
(b) $(2K - 1)$-digit accuracy for infinitely many n.

Proof. Because K is the number of digits in q_n, we have

$$10^{K-1} \le q_n < 10^K.$$

The denominators q_n in a continued fraction increase with n, so $q_{n+1} \geq 10^{K-1}$ and inequality (1) implies

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{10^{2K-2}}, \tag{2}$$

which proves (a). To prove (b) we note that there are infinitely many n for which the number of digits in the denominator q_{n+1} is greater than the number of digits in q_n (otherwise the q_n would not increase with n). Hence for infinitely many n we have $q_n = 10^{K-1}r$ and $q_{n+1} = 10^K R$, where $r \geq 1$ and $R \geq 1$. Consequently $q_n q_{n+1} = (rR)10^{2K-1}$ for these n, and inequality (1) becomes

$$\left| \alpha - \frac{p_n}{q_n} \right| < \frac{1}{(rR)10^{2K-1}}, \tag{3}$$

which proves (b) because $rR \geq 1$. ■

If $rR \geq 10$, which can happen in examples chosen at random, inequality (3) shows that p_n/q_n will represent α with at least $2K$-digit accuracy. Many such examples exist, some of which were noted above. It is also possible to construct examples that agree in more than $2K$ digits. For example, the third convergent of the continued fraction expansion of the irrational number

$$\alpha = 10(11 - 13\log 7) = 0.137254798000\ldots$$

is $7/51 = 0.13725490196\ldots.$, which agrees with α to 6-digit accuracy, even though its denominator only has two digits.

The worst possible case occurs when the product rR is close to 1. An example is the continued fraction for $\alpha = 1/(\sqrt{26} - 4)$. To obtain this continued fraction note that the irrational number $x = \sqrt{26} - 4$ satisfies the quadratic equation $(x-1)(x+9) = 1$, so $x - 1 = 1/(x+9) = 1/(10 + (x-1))$. Repeated use of this relation gives the infinite continued fraction $x - 1 = [0; 10, 10, 10, \ldots]$ with $a_n = 10$ for all $n \geq 1$, from which we find $x = [1; 10, 10, 10, \ldots]$ and $\alpha = 1/x = [0; 1, 10, 10, 10, \ldots]$, whose first seven successive denominators q_n turn out to be 1, 11, 111, 1121, 11321, 114331, 1154631. In this example the number of digits in q_{n+1} is exactly one more than the number in q_n for every n. The decimal representations of p_n/q_n and α agree in exactly $2K - 1$ digits, where K is the number of digits in q_n. This example shows that in general you cannot do better than $(2K - 1)$ digit accuracy for all n.

Finally, the reader can verify that base ten is not essential. A similar theorem holds for numbers expressed in any base.

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Running with Rover

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It all started one day when I went running on a trail with my faithful dog Rover. Now Rover does not actually rove. In fact, Rover is so well trained that he always runs exactly one yard to my right. As long as I change direction smoothly, he will adjust his speed and his path perfectly so as to remain in this position. Of course, I must choose my path so that he is not required to run through trees or other obstacles.

On this particular day our trail was flat but curvy. We looped around several times, as shown in FIGURE 1. As nearly as I can tell from a map, the farthest we got from our starting and finishing point S was about a mile as the crow flies (not as the Crofoot runs).

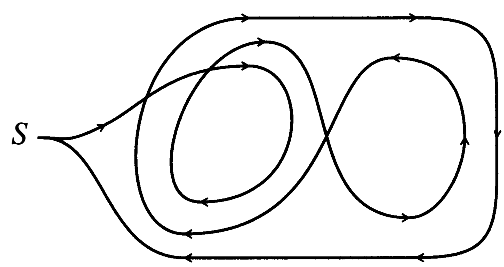

Figure 1 The trail

Prostrate on the couch at the end of our run, I could not help but notice that Rover still had lots of energy. This upset my athletic self-image to the extent that I almost resolved to cut back on ice cream and begin a rigorous program of monthly exercise. Fortunately, I was saved from this fanaticism by the reassuring realization that I had run farther than Rover, having been on the outside on most of our turns.

Now I would like to know this: How much farther did I run than Rover?

Surprisingly, the exact answer to this question can be determined from FIGURE 1 without any additional information. In the course of arriving at that answer, we will develop some ideas that are central to differential geometry and topology. (Expert readers who already know the answer may be interested to know that this elementary treatment does not require using arc length as a new parameter. Switching between two independent variables, time and arc length, is often a stumbling block for students.)

Since FIGURE 1 is quite complicated, let's consider a simpler situation first. Suppose that I run with Rover on a track instead of a trail. The track has the usual shape, with semicircular ends connected by straight sides. As we round the ends, the outside runner follows a circle of radius $R + 1$ while the inside runner follows a circle of radius R. During one complete lap, the difference in total distance is $2\pi(R + 1) - 2\pi R = 2\pi$ yards. Note that this difference does not involve R. So the length of the track does not matter.

Now we start to wonder (you and I, that is—not Rover!): since the length of the track does not matter, perhaps the shape of the track does not matter. The difference in distance will be greatest along stretches where the track turns most quickly, but it seems plausible that the difference in total distance depends only on the total amount of turning that occurs along the way, taking account of the direction of the turns. We will show that this is indeed the case.

The central idea is that smooth curves can be approximated by arcs of circles. What this means physically is that when I am running on a curved path, both my direction and the sharpness of my turn at any moment are the same as if I were running on some circular path of an appropriate radius. This approximating circle at a point on my path is called the *osculating circle* at the point, and its radius is the *radius of curvature* of my path at the point. As I run with Rover along a curved trail, the difference in our speeds at any time is related to the different radii of our osculating circles. In order to maintain his position alongside of me, Rover must adjust his speed and direction to ensure that at each instant his osculating circle has a radius one yard larger or smaller than mine, depending on the direction of our turn.

Curves and angle functions The position of an object moving in a plane (a running person or dog, for example) can be described relative to some fixed reference point O by a vector-valued function $\mathbf{r}(t) = \langle x(t), y(t)\rangle$, where t is time. This function is a *parametrization* for an *oriented curve* Γ. FIGURE 2 shows such a curve, parametrized by $\mathbf{r}(t)$ as t varies within an interval $[a, b]$. The arrows on the curve indicate the direction along the curve corresponding to increasing t.

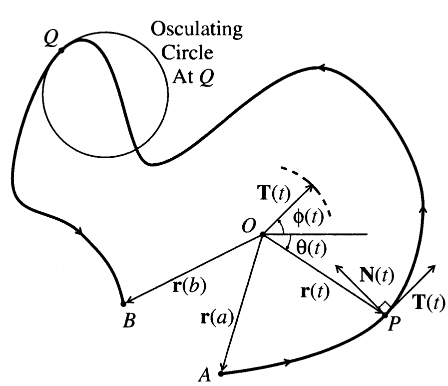

Figure 2 Analysis of a Curve

If $\mathbf{r}(t)$ is differentiable, the derivative $\mathbf{r}'(t) = \langle x'(t), y'(t)\rangle$ represents the *velocity* of an object whose position is given by $\mathbf{r}(t)$. For any t such that $\mathbf{r}'(t) \neq \mathbf{0}$, the vector $\mathbf{r}'(t)$ is tangent to Γ at the point $(x(t), y(t))$ and points in the direction of Γ. The magnitude of this vector is the speed, which can be integrated to calculate the distance travelled along Γ.

The vector $\langle -y'(t), x'(t)\rangle$, obtained by rotating the velocity vector counter-clockwise 90 degrees, is normal to Γ at the same point. Dividing by the magnitudes of these vectors, we obtain a *unit tangent vector* $\mathbf{T}(t)$ and a *unit normal vector* $\mathbf{N}(t)$:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\,\mathbf{r}'(t)\,|} = \frac{\langle x'(t), y'(t)\rangle}{|\,\mathbf{r}'(t)\,|}, \tag{1}$$

$$\mathbf{N}(t) = \frac{\langle -y'(t), x'(t)\rangle}{|\,\mathbf{r}'(t)\,|}. \tag{2}$$

These vectors are shown at the point P in FIGURE 2.

Let Γ be a curve parametrized by a continuous function $\mathbf{r}(t)$. Assume that the origin does not lie on Γ, so that $\mathbf{r}(t)$ is never the zero vector. For any particular value of t, the point $(x(t), y(t))$ on Γ can be represented using polar coordinates $(r(t), \theta(t))$, where

$$r(t) = \sqrt{[x(t)]^2 + [y(t)]^2} = |\,\mathbf{r}(t)\,|, \tag{3}$$

$$\cos\theta(t) = \frac{x(t)}{r(t)}, \qquad \sin\theta(t) = \frac{y(t)}{r(t)}. \tag{4}$$

The angle $\theta(t)$ is not uniquely defined by these equations. If $\theta_0(t)$ is one solution of the equations, then the general solution is $\theta(t) = \theta_0(t) + 2\pi n(t)$, where $n(t)$ is any integer-valued function of t. However, once we have selected an angle from among the various possible angles at any particular point on the curve, we can change this angle continuously as we move along the curve, never jumping by a multiple of 2π. The resulting *continuous angle function* $\theta(t)$ is not uniquely determined by the parametrization $\mathbf{r}(t)$, but any two continuous angle functions differ by a constant, which is an integer multiple of 2π. This informal discussion of continuous angle functions is made rigorous elsewhere [**1**; **3**, pp. 17–19, 35–37]. From now on, whenever we talk about an angle function for a curve, we will always mean a continuous angle function—a continuous function $\theta(t)$ such that equations (4) are satisfied for all t.

As t varies from $t = t_1$ to $t = t_2$, the change $\theta(t_2) - \theta(t_1)$ does not depend on the particular angle function $\theta(t)$, since any two angle functions differ by a constant. This change can be interpreted as the total angle through which the position vector $\mathbf{r}(t)$ turns as t varies from t_1 to t_2. In particular, $\theta(b) - \theta(a)$ represents the total angle through which $\mathbf{r}(t)$ turns along the entire curve Γ. By the total angle we mean the total *signed* angle, which increases as $\mathbf{r}(t)$ turns counterclockwise and decreases as $\mathbf{r}(t)$ turns clockwise, possibly taking negative values.

If $\mathbf{r}(a) = \mathbf{r}(b)$, then Γ is a *closed* curve. In this case the change $\theta(b) - \theta(a)$ must be an integer multiple of 2π because the angles $\theta(a)$ and $\theta(b)$ refer to the same point. The integer $[\theta(b) - \theta(a)]/(2\pi)$ represents the number of times that Γ winds around the origin (with counterclockwise counting as positive and clockwise counting as negative). This integer is called the *index*, or *winding number*, of Γ with respect to the origin.

If $\mathbf{r}(t)$ is differentiable, then any angle function $\theta(t)$ has a derivative, which can be calculated from equations (3) and (4):

$$\theta'(t) = \frac{x(t)y'(t) - x'(t)y(t)}{|\,\mathbf{r}(t)\,|^2} = \frac{x(t)y'(t) - x'(t)y(t)}{[x(t)]^2 + [y(t)]^2}. \tag{5}$$

The change in angle along the curve can be computed by integration:

$$\theta(b) - \theta(a) = \int_a^b \theta'(t)\,dt.$$

Having talked about angle functions, we should give some indication of how we intend to apply them. An angle function for a curve describes the turning of the position vector $\mathbf{r}(t)$. For the curve in FIGURE 1, what concerns us is not the turning of $\mathbf{r}(t)$ but the turning of a *tangent vector*. Therefore, in order to apply the idea of an angle function, we will need to consider a curve constructed from the given curve by using the tangent vector $\mathbf{T}(t)$ of the given curve as the parametrization for the new curve. This idea is developed in the next section.

Turning rate along a curve The unit tangent vector $\mathbf{T}(t)$ is defined at all points on Γ where the velocity $\mathbf{r}'(t)$ is not zero. We now demand that $\mathbf{T}(t)$ be a continuous function defined on the entire parameter interval $[a, b]$. This will happen when the following two conditions are satisfied:

1. The function $\mathbf{r}(t)$ is continuously differentiable on $[a, b]$ (using one-sided derivatives at the endpoints a and b),
2. At any point where $\mathbf{r}'(t) = 0$, there is no change in direction of the curve, so that the function $\mathbf{T}(t)$ can be continuously extended to include this point in its domain.

Then $\mathbf{T}(t)$ may be regarded as a parametrization for a curve Γ' called the *tangent indicatrix.* The points of Γ' all lie on the unit circle. The vector $\mathbf{T}(t)$ may trace out parts of this circle repeatedly as t varies, in which case these repetitions are considered to be distinct parts of the parametrized curve Γ'.

Let $\phi(t)$ denote a continuous angle function for Γ'. We want to apply equation (5) to obtain a formula for $\phi'(t)$. The position vector for Γ' is $\mathbf{T}(t) = c(t)\mathbf{r}'(t)$, where $c(t) = 1/|\mathbf{r}'(t)|$. Therefore, in equation (5) we will replace $x(t)$ by $c(t)x'(t)$ and $y(t)$ by $c(t)y'(t)$). Assuming that $x''(t)$ and $y''(t)$ both exist for all t, we calculate

$$\phi'(t) = \frac{x'(t)y''(t) - x''(t)y'(t)}{|\mathbf{r}'(t)|^2}. \tag{6}$$

Actually this formula can be derived assuming only that $c(t)$ is *any* positive, differentiable function, not necessarily $1/|\mathbf{r}'(t)|$. Intuitively this is because the amount of turning of a tangent vector does not depend on the length of the tangent vector. For purposes of defining the angle function $\phi(t)$, we could just as well use a curve parametrized simply by $\mathbf{r}'(t)$ instead of $\mathbf{T}(t)$.

A straightforward calculation, starting from equations (1) and (2) and using (6), yields the following formulas for the derivatives of the unit tangent vector and the unit normal vector:

$$\mathbf{T}'(t) = \phi'(t)\mathbf{N}(t), \qquad \mathbf{N}'(t) = -\phi'(t)\mathbf{T}(t). \tag{7}$$

Taking magnitudes here, and recalling that $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are unit vectors, we see that $|\mathbf{T}'(t)| = |\mathbf{N}'(t)| = |\phi'(t)|$. Thus the magnitude of the rate of change of the angle $\phi(t)$ is the same as the magnitude of the rate of change of the unit tangent vector and the unit normal vector. All three of these rates are equivalent measures of the rate of turning along Γ.

The rate of turning is related to the radius of curvature. Most calculus books derive the following expression for the radius of curvature at any point $(x(t), y(t))$ on Γ:

$$\frac{([x'(t)]^2 + [y'(t)]^2)^{3/2}}{|x'(t)y''(t) - x''(t)y'(t)|}.$$

Using equation (6), we can express this as $|\mathbf{r}'(t)|/|\phi'(t)|$. The denominator here may be zero for some particular value of t, in which case the radius of curvature at the corresponding point on the curve Γ is considered to be either undefined or infinite. This happens at all points if the curve is a straight line. It also happens at points where the curve changes turning direction, and, for an instant, is not curving at all.

If we omit the absolute value in the denominator in the above expressions for the radius of curvature, we obtain a *signed* radius of curvature, which is positive when the curve is turning counterclockwise and negative when it is turning clockwise. Letting $R(t)$ denote the signed radius of curvature at the point with position vector $\mathbf{r}(t)$, we have

$$R(t) = \frac{|\mathbf{r}'(t)|}{\phi'(t)}. \tag{8}$$

Back to Rover Now consider Rover and me as we run along the trail shown in FIGURE 1. Choose a convenient coordinate system, and let $\mathbf{r}_1(t)$ be a parametrization for my path. Define unit tangent and unit normal vectors as before, $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

Assuming that Rover runs exactly α yards to my right, where α is a constant, Rover's path is parametrized by

$$\mathbf{r}_2(t) = \mathbf{r}_1(t) - \alpha\mathbf{N}(t).$$

Taking the derivative and using equations (1) and (7), we get

$$\begin{aligned}\mathbf{r}_2'(t) &= \mathbf{r}_1'(t) - \alpha\mathbf{N}'(t) = \mathbf{r}_1'(t) + \alpha\phi'(t)\mathbf{T}(t)\\ &= [\,|\,\mathbf{r}_1'(t)\,| + \alpha\phi'(t)]\mathbf{T}(t).\end{aligned}$$

Then, since $\mathbf{T}(t)$ is a unit vector,

$$|\,\mathbf{r}_2'(t)\,| = \big|\,|\,\mathbf{r}_1'(t)| + \alpha\phi'(t)\,\big| = |\,\mathbf{r}_1'(t)\,| + \alpha\phi'(t),$$

provided the quantity $|\,\mathbf{r}_1'(t)\,| + \alpha\phi'(t)$ is never negative. This is a reasonable assumption, as we will see shortly. First we will complete our calculation. Using this equation and the standard formula for arc length, we calculate the difference in arc length along the two paths:

$$\begin{aligned}L_2 - L_1 &= \int_a^b |\,\mathbf{r}_2'(t)\,|\,dt - \int_a^b |\,\mathbf{r}_1'(t)\,|\,dt\\ &= \alpha\int_a^b \phi'(t)\,dt = \alpha[\phi(b) - \phi(a)].\end{aligned}$$

As explained earlier, the quantity $\phi(b) - \phi(a)$ represents the total angle through which the tangent vector $\mathbf{T}(t)$ turns as t varies from a to b. We can determine this angle just by looking at FIGURE 1. The two loops in the middle can be ignored, because the clockwise loop cancels the counterclockwise loop. The remaining part of the curve amounts to one-and-a-half turns clockwise. (Note that we finish our run going in the opposite direction from our starting direction.) Thus the total turning angle is -3π, and $L_2 - L_1 = \alpha(-3\pi)$. Taking $\alpha = 1$ yard, we conclude that I ran 3π yards farther than Rover. No wonder I was so tired!

In our calculation we assumed that $|\,\mathbf{r}_1'(t)\,| + \alpha\phi'(t) \geq 0$ for all t. Suppose this were violated, so that $|\,\mathbf{r}_1'(t)\,| + \alpha\phi'(t) < 0$ at some time t. Letting $R_1(t)$ be the signed radius of curvature of my path, and using equation (8), it would follow that $-\alpha < R_1(t) < 0$. This would mean that my path was turning clockwise with a radius of curvature $|\,R_1(t)\,| < \alpha$. I would be turning *towards* Rover so sharply that he could not compensate by slowing down. Instead he would have to do some additional running around (perhaps on a rather small scale) in order to remain in the ideal position beside me. Our simple formula for $L_2 - L_1$ would no longer apply. The reader might like to think about what happens if, for example, I run clockwise around a circle of radius less than one yard while Rover remains exactly one yard to my right.

We have been using real vector notation for our parametrizations, but complex notation has its advantages. The reader is invited to investigate how much simpler the computations become when my path is parametrized by a complex-valued function $\zeta_1(t)$. For instance, our unit normal vector is simply $ie^{i\phi(t)}$, where $\phi(t)$, defined as before, is an argument function for $\zeta_1'(t)$.

From a purely mathematical point of view, we have been comparing the arc lengths of two *parallel curves*. The mathematics we have presented is certainly not new, but there seems to be no single reference from which it can be easily extracted. Many of the ideas are to be found scattered through the first 25 pages of Klingenberg [**4**], while Exercise 6 on p. 47 of do Carmo [**2**] states a special case of our formula for $L_2 - L_1$. Many excellent sources are available [**5**, **6**] for anyone interested in delving further into differential geometry.

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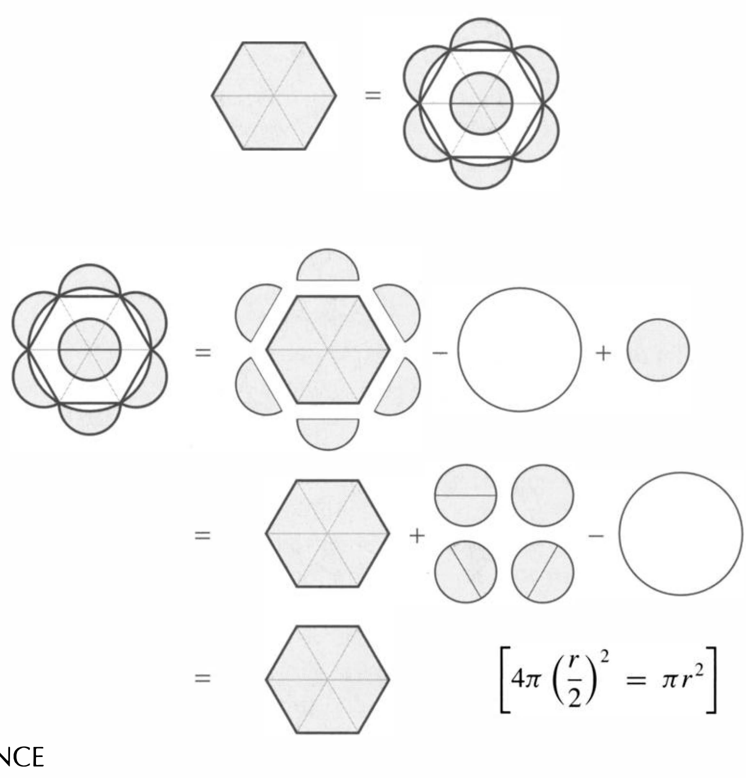
1. John A. Baker, Plane curves, polar coordinates and winding numbers, this MAGAZINE **64** (1991), 75–91.
2. Manfredo P. do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice-Hall, Englewood Cliffs, NJ, 1976.
3. William Fulton, *Algebraic Topology: A First Course*, Springer, New York, 1995.
4. Wilhelm Klingenberg, *A Course in Differential Geometry*, Springer, New York, 1978.
5. Barrett O'Neill, *Elementary Differential Geometry*, 2nd ed., Academic Press, San Diego, 1997.
6. Michael Spivak, *A Comprehensive Introduction to Differential Geometry*, 3rd ed., Vol. 1, Publish or Perish, Houston, 1999.

Proof Without Words: Lunes and the Regular Hexagon

THEOREM. If a regular hexagon is inscribed in a circle and six semicircles constructed on its sides, then the area of the hexagon equals the area of the six lunes plus the area of a circle whose diameter is equal in length to one of the sides of the hexagon. [Hippocrates of Chios, ca. 440 B.C.E]

Proof.



$$\left[4\pi\left(\frac{r}{2}\right)^2 = \pi r^2\right]$$

REFERENCE

1. William Dunham, *Journey through Genius*, John Wiley and Sons, New York, 1990, Chapter 1.

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PROBLEMS

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Proposals

To be considered for publication, solutions should be received by March 1, 2003.

1653. *Proposed by Larry Hoehn, Austin Peay State University, Clarksville, TN.*

In $\triangle ABC$, let F and G be points on $\overline{AB}$ with F between G and A, and let D and E be on $\overline{AC}$ with E between D and A. Prove that if $\triangle BDE \sim \triangle CGF$, then quadrilateral $DEFG$ is a trapezoid.

1654. *Proposed by David Singmaster, London, England.*

A salesman's office is located on a straight road. His N customers are all located along this road to the east of the office, with the office of customer k at distance k from the salesman's office. The salesman must make a driving trip whereby he leaves the office, visits each customer exactly once, then returns to the office. Because he makes a profit on his mileage allowance, the salesman wants to drive as far as possible during his trip. What is the maximum possible distance he can travel on such a trip, and how many different such trips are there? Assume that if the travel plans call for the salesman to visit customer j immediately after he visits customer i, then he drives directly from i to j.

1655. *Proposed by Costas Efthimiou, Department of Physics, University of Central Florida, Orlando, FL.*

Let $0 < a, b, c < 1$ with $ab + bc + ca = 1$. Prove that

$$\frac{a}{1-a^2} + \frac{b}{1-b^2} + \frac{c}{1-c^2} \geq \frac{3\sqrt{3}}{2},$$

and give the conditions under which equality holds.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution.

Solutions should be written in a style appropriate for this MAGAZINE. Each solution should begin on a separate sheet.

Solutions and new proposals should be mailed to Elgin Johnston, Problems Editor, Department of Mathematics, Iowa State University, Ames, IA 50011, or mailed electronically (ideally as a LaTeX file) to `ehjohnst@iastate.edu`. All communications should include the readers name, full address, and an e-mail address and/or FAX number.

1656. *Proposed by David M. Bloom, Hartsdale, NY.*

Let $a_1, a_2, \ldots, a_k$, $k \geq 2$ be pairwise relatively prime positive integers, and let

$$b_j = \left(\prod_{i=1}^{k} a_i\right) \Big/ a_j, \quad 1 \leq j \leq k,$$

and

$$M = (k-1)a_1 a_2 \cdots a_k - (b_1 + b_2 + \cdots + b_k).$$

Prove that for each positive integer n, one and only one of the integers n and $M - n$ can be expressed in the form $x_1 b_1 + x_2 b_2 + \cdots + x_k b_k$, where the x_i are non-negative integers. (*Note:* This problem generalizes Problem 1561, this MAGAZINE, Dec. 1999, 411–412.)

1647. *Proposed by Leroy Quet, Denver, CO.*

Prove that for all $x > 0$,

$$\sum_{j=1}^{\infty} \sum_{k=1}^{j} \frac{x^{(j+1)^{-2}} - x^{k^{-2}}}{(j+1)^2 - k^2} = \frac{\pi^2}{8} - \frac{3}{4} \sum_{j=1}^{\infty} \frac{x^{j^{-2}}}{j^2}.$$

(This problem was originally published with a typographical error in the displayed equation. The error has been corrected in the statement above.)

Quickies

Answers to the Quickies are on page 323.

Q923. *Proposed by Murray Klamkin, University of Alberta, Edmonton, AB, Canada.*

Let x_k, y_k, $1 \leq k \leq n$ be positive real numbers and let r, s be real numbers with $\sum_{k=1}^{n} x_k^r = \sum_{k=1}^{n} y_k^s = 1$. Prove that if $\sum_{k=1}^{n} (x_k^{r(m+1)} / y_k^{sm}) = 1$ for some positive integer m, then $\sum_{k=1}^{n} (x_k^{r(m+1)} / y_k^{sm}) = 1$ for every positive integer m.

Q924. *Proposed by José Luis Díaz-Barrero and Juan José Egozcue, Barcelona, Spain.*

Let α, β, and γ be real numbers. Prove that

$$\left(1 + \frac{\cos^2\alpha + \cos^2\beta}{2}\right)\left(1 + \frac{\cos^2\beta + \cos^2\gamma}{2}\right)\left(1 + \frac{\cos^2\gamma + \cos^2\alpha}{2}\right)$$
$$\geq 8 \sin^4\left(\frac{\alpha}{2}\right) \sin^4\left(\frac{\beta}{2}\right) \sin^4\left(\frac{\gamma}{2}\right),$$

and describe the conditions under which equality holds.

Solutions

A Recursive Functional Equation **October 2001**

1628. *Proposed by Răzvan Gelca, Texas Tech University, Lubbock, TX.*

Find all functions $f: \mathbb{N} \longrightarrow \mathbb{N}$ such that, for all $n \in \mathbb{N}$,

$$f(f(f(n))) + 6f(n) = 3f(f(n)) + 4n + 2001.$$

Solution by Li Zhou, Polk Community College, Winter Haven, FL.

It is easy to check that $f(n) = n + 667$ is one solution; we show that it is the only solution. For a given $n \in \mathbb{N}$, the given equation implies that

$$a_{k+3} - 3a_{k+2} + 6a_{k+1} - 4a_k = 2001, \qquad k \geq 0,$$

where $a_k = f^{(k)}(n)$ and $a_0 = n$. Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then

$$(1 - 3x + 6x^2 - 4x^3)G(x) = a_0 + (a_1 - 3a_0)x + (a_2 - 3a_1 + 6a_0)x^2 + \frac{2001x^3}{1-x}.$$

Because $1 - 3x^2 + 6x^2 - 4x^3 = (1-x)(1-2x+4x^2) = \frac{(1-x)(1+8x^3)}{1+2x}$, we have

$$G(x) = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{Cx + D}{1 - 2x + 4x^2}$$
$$= \sum_{k=0}^{\infty} Ax^k + \sum_{k=0}^{\infty} B(k+1)x^k + (Cx + D)(1 + 2x) \sum_{k=0}^{\infty} (-2x)^{3k}.$$

Because $a_k = f^{(k)}(n) > 0$ for all k, we must have $C = D = 0$. Consequently

$$\left(a_0 + (a_1 - 3a_0)x + (a_2 - 3a_1 + 6a_0)x^2\right)(1-x) + 2001x^3$$
$$= (A(1-x) + B)(1 - 2x + 4x^2).$$

Comparing coefficients, we find that $a_0 = A + B$ and $a_1 - 4a_0 = -3A - 2B$. Setting $x = 1$, we find that $2001 = 3B$. Hence, $a_1 - a_0 = B = 667$, and $f(n) = a_1 = a_0 + 667 = n + 667$.

Also solved by JPV Abad, Michel Bataille (France), Alexander Blokh, Con Amore Problem Group (Denmark), Knut Dale (Norway), Robert L. Doucette, Daniele Donini (Italy), Fejentalaltuka Szeged Problem Solving Semigroup (Hungary), Natalio H. Guersenzvaig (Argentina), Kee-Wai Lau (Hong Kong), Kim McInturff, Michael Reid, Rolf Richberg (Germany), Heinz-Jürgen Seiffert (Germany), and the proposer. One solution with no name and two incomplete submissions were also received.

Sines and Derivatives **October 2001**

1629. *Proposed by Peng Gao, Ann Arbor, MI.*

Define $f_1(x) = \frac{d}{dx}\sin^2 x$, and for integer $n \geq 2$ define,

$$f_n(x) = \frac{d}{dx}\left((\sin^2 x) f_{n-1}(x)\right).$$

Find the value of $f_n(\pi/2)$.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada.

We can write

$$f_n(x) = \left(D \sin^2 x\right)^n = \sin^{-2} x \left((\sin^2 x)D\right)^n \sin^2 x,$$

where $D = d/dx$. Next make the substitution $t = \cot x$ and note that $(\sin^2 x)D = -d/dt$. Hence

$$f_n(x) = (-1)^n \sin^{-2} x \frac{d^n}{dt^n}\left(\frac{1}{1+t^2}\right) = \frac{1}{2}(-1)^n \sin^{-2} x \frac{d^n}{dt^n}\left(\frac{1}{1+it} + \frac{1}{1-it}\right)$$

$$= \frac{1}{2} n! \sin^{-2} x \left(\frac{i^n}{(1+it)^{n+1}} + \frac{(-i)^n}{(1-it)^{n+1}} \right)$$

$$= \frac{1}{2} n! \sin^{-2} x \left(\frac{i^n}{(i\sqrt{1+t^2}e^{-ix})^{n+1}} + \frac{(-i)^n}{(-i\sqrt{1+t^2}e^{ix})^{n+1}} \right)$$

$$= n!(\sin^{n-1} x) \sin((n+1)x) .$$

Hence $f_{2k+1}(\pi/2) = 0$ and $f_{2k}(\pi/2) = (-1)^k(2k)!$.

Note: Many readers used induction to establish $f_n(x) = n!(\sin^{n-1} x) \sin((n+1)x)$.

Also solved by JPV Abad, Michel Bataille (France), J. C. Binz (Switzerland), Con Amore Problem Group (Denmark), Charles K. Cook, Joseph Coster (Luxembourg), Knut Dale (Norway), Charles R. Diminnie, Daniele Donini (Italy), Robert L. Doucette, Steve Edwards, Fejentalaltuka Szeged Problem Solving Semigroup (Hungary), James C. Hickman, Peter M. Jarvis, Elias Lampakis (Greece), Reiner Martin, Kim McInturff, Richard F. Melka and Yong Z. Chen, Greg Neumer, Rolf Richberg (Germany), Sam L. Robinson and Gerald Thompson, Earl A. Smith, Irving C. Tang, Nora S. Thornber, Michael Vowe (Switzerland), Michael Woltermann, Li Zhou, and the proposer.

A Centroidal Equality **October 2001**

1630. *Proposed by Geoffrey A. Kandall, Hamden, CT.*

Let P be in the interior of $\triangle ABC$, and let lines AP, BP, and CP intersect sides BC, CA, and AB in L, M, and N, respectively. Prove that if

$$\frac{AP}{PL} + \frac{BP}{PM} + \frac{CP}{PN} = 6,$$

then P is the centroid of $\triangle ABC$.

Solution by Murray S. Klamkin, University of Alberta, Edmonton, AB, Canada.

We prove the following generalization:

Let P be a point in the interior of the n-dimensional simplex $A_0A_1A_2\ldots A_n$, and for $0 \le k \le n$ let the cevian from A_k through P intersect the opposite face in B_k. If

$$\sum_{k=0}^{n} \frac{A_kP}{PB_k} = n(n+1),$$

then P is the centroid of the simplex.

We use barycentric coordinates. Let $\mathbf{P}$, $\mathbf{A}_k$, and $\mathbf{B}_k$ denote vectors from a common origin to the points P, A_k, and B_k respectively. Then $\mathbf{P} = \sum_{k=0}^{n} x_k\mathbf{A}_k$ where $\sum_{k=0}^{n} x_k = 1$ and each $x_k > 0$, and $\mathbf{B}_k = (\mathbf{P} - x_k\mathbf{A}_k)/(1 - x_k)$. We then have

$$A_kP = \|\mathbf{A}_k - \mathbf{P}\|,$$

$$A_kB_k = \|\mathbf{A}_k - (\mathbf{P} - x_k\mathbf{A}_k)/(1-x_k)\| = \|\mathbf{A}_k - \mathbf{P}\|/(1-x_k),$$

and

$$B_kP = A_kB_k - A_kP = x_k\|\mathbf{A}_k - \mathbf{P}\|/(1-x_k).$$

It follows that

$$n(n+1) = \sum_{k=0}^{n} \frac{A_kP}{PB_k} = \sum_{k=0}^{n} \left(\frac{1}{x_k} - 1 \right),$$

so $\sum_{k=0}^{n} 1/x_k = (n+1)^2$. However, by the Cauchy-Schwarz inequality,

$$\sum_{k=0}^{n} \frac{1}{x_k} = \left(\sum_{k=0}^{n} x_k\right)\left(\sum_{k=0}^{n} \frac{1}{x_k}\right) \geq (n+1)^2,$$

with equality if and only if $x_k = 1/(n+1)$, $0 \leq k \leq n$. It follows that P is the centroid of the simplex.

This completes the solution of the problem. However, other similar results also hold. Indeed, it also follows that if any of the following three equations holds, then P is the centroid of the simplex:

$$\sum_{k=0}^{n} \frac{PB_k}{A_kP} = \frac{n+1}{n}, \qquad \sum_{k=0}^{n} \frac{A_kB_k}{A_kP} = \frac{(n+1)^2}{n}, \qquad \sum_{k=0}^{n} \frac{A_kB_k}{PB_k} = (n+1)^2.$$

If the first of these equations is true, then

$$\frac{n+1}{n} = \sum_{k=0}^{n} \frac{PB_k}{A_kP} = \sum_{k=0}^{n} \frac{x_k}{1-x_k} = -(n+1) + \sum_{k=0}^{n} \frac{1}{1-x_k},$$

so $\sum_{k=0}^{n} 1/(1-x_k) = (n+1)^2/n$. By the Cauchy-Schwarz inequality,

$$\sum_{k=0}^{n} 1/(1-x_k) \geq (n+1)^2 \Big/ \sum_{k=0}^{n} (1-x_k) = (n+1)^2/n,$$

with equality if and only if $x_k = 1/(n+1)$, $0 \leq k \leq n$. Similar arguments can be used to show that if either of the other two equations is true, then P is the centroid of the simplex.

Note: Miguel Amengual Covas of Spain points out that the triangle version of this problem appeared on a Romanian Mathematics Competition. See Revista de Matematica din Timisora, Annul II (seriea a 4-a), nr. 1-1997, pp. 16–17. The triangle version of this problem also appeared as problem E 1043 in the *American Mathematical Monthly*, **59** (1952), pg. 697, with solution in **60** (1953), pg. 421.

Also solved by JPV Abad, Herb Bailey, Roy Barbara (Lebanon), Michel Bataille (France), J. C. Binz (Switzerland), Pierre Bornsztein (France), Gerald D. Brown, Scott H. Brown, Minh Can, Timothy V. Craine, Knut Dale (Norway), Daniele Donini (Italy), Robert L. Doucette, Petar D. Drianov (Canada), Mordechai Falkowitz (Canada), Fejentalaltuka Szeged Problem Solving Semigroup (Hungary), Ovidiu Furdui, Michael Golomb, H. Guggenheimer, Peter Hohler (Switzerland), Samee Ullah Khan (Hong Kong), Ken Korbin, Victor Y. Kutsenok, Elias Lampakis (Greece), Matti Lehtinen (Finland), P. E. Nüesch (Switzerland), Heinz-Jürgen Seiffert (Germany), Achilleas Sinefakopoulos, Raul A. Simon (Chile), Alexey Vorobyov, Michael Vowe (Switzerland), Homer White, Michael Woltermann, Peter Y. Woo, Li Zhou, and the proposer.

Tiling a Deleted Checkerboard **October 2001**

1631. *Proposed by Hoe-Teck Wee, student, Massachusetts Institute of Technology, Cambridge, MA.*

For each integer $n > 3$ and not divisible by 3, how many ways are there to delete a square from an $n \times n$ chess board so that the remaining board can be tiled with 3×1 trominos?

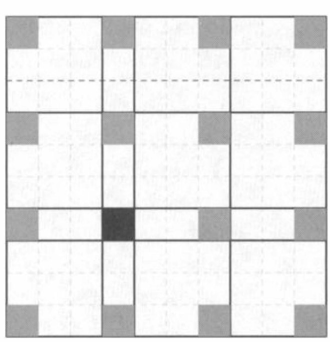
Solution by Philip D. Straffin, Beloit College, Beloit, WI.

The number of ways to delete one square so the remaining board is tilable is $((n+2)/3)^2$ if $n \equiv 1 (\text{mod}\, 3)$, and $((n-2)/3)^2$ if $n \equiv 2 (\text{mod}\, 3)$.

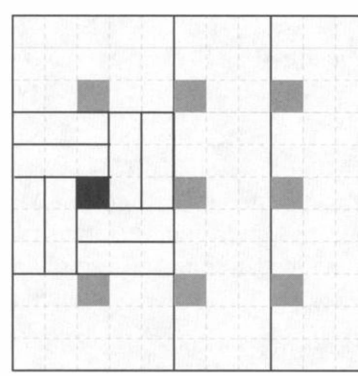
Counting from the bottom and the left, let (i, j) denote the square in the ith row and the jth column. Label square (i, j) by $k_{i,j} = 0, 1$, or 2, where $k_{i,j} \equiv i - j \pmod 3$. In any partial tiling of the board by 3×1 trominos, each tromino covers three squares with three different labels. Hence in the tilings, the same number of 0s, 1s, and 2s must be covered. The board has $(n^2 + 2)/3$ squares labeled 0, but only $(n^2 - 1)/3$ squares labeled each of 1 and 2. Thus, if the board with one square deleted can be tiled, the deleted square (i, j) must satisfy $i - j \equiv 0 \pmod 3$.

Next label square (i, j) by $m_{i,j} = 0, 1$, or 2, where $m_{i,j} \equiv i + j \pmod 3$. By reasoning similar to that presented above, the deleted square (i, j) must also satisfy $i + j \equiv n + 1 \pmod 3$. Thus, if $n \equiv 1 \pmod 3$, then the deleted square must satisfy $i - j \equiv 0 \pmod 3$ and $i + j \equiv 2 \pmod 3$, that is $i \equiv j \equiv 1 \pmod 3$. There are $((n + 2)/3)^2$ such squares (i, j). If $n \equiv 2 \pmod 3$, then the deleted square must satisfy $i - j \equiv 0 \pmod 3$ and $i + j \equiv 0 \pmod 3$, that is, $i \equiv j \equiv 0 \pmod 3$. There are $((n - 2)/3)^2$ such squares.

It remains to show that if a square of the proper form is deleted, then the resulting board can indeed be tiled with 3×1 trominos. If $n \equiv 1 \pmod 3$ and the square $(i, j) = (3p + 1, 3q + 1)$ is deleted, then the rest of the ith row and the rest of the jth column can be tiled with 3×1 trominos, and the remainder of the board can be decomposed into tilable 3×3 squares. The situation for $n = 10$ is shown below. If $n \equiv 2 \pmod 3$ and square $(i, j) = (3p, 3q)$ is deleted, then the 5×5 square centered at (i, j) and with the center square deleted can be tiled in a "pinwheel" pattern, and the remainder of the board can be decomposed into tilable $3 \times l$ rectangles. The situation for $n = 11$ is shown below.



$n = 10$



$n = 11$

■ deleted square   ▒ other deletable squares

Also solved by JPV Abad, Pierre Bornsztein (France), Keith Chavey, Knut Dale (Norway), Robert L. Doucette, Fejentalaltuka Szeged Problem Solving Semigroup (Hungary), Marty Getz and Dixon Jones, Mark Kidwell, Murray S. Klamkin (Canada), Reiner Martin, Jonathan Nilsson, Alexey Vorobyov, Michael Woltermann, Li Zhou, and the proposer. There was one solution with no name.

Common Divisors **October 2001**

1632. *Proposed by Erwin Just, Bronx Community College, New York, NY.*

Prove that there are an infinite number of integers n for which there exists a set of n distinct positive odd integers such that each member of the set divides the sum of all the members of the set.

The following solution was provided by several of the readers listed below.

We use induction to produce a sequence of sets with the desired property, where each set in the sequence has more elements than the preceding set. First observe that the set

$$A_1 = \{1, 21, 45, 63, 75, 105, 225, 315, 525\}$$

is a set with the desired property and that because A_1 has an odd number of elements, the sum of the elements of A_1 is also odd. Next assume that $A_n = \{a_1, a_2, \ldots, a_{m_n}\}$, with m_n odd, is a set of distinct odd integers, that the sum of the elements of S is s_n, and that $a_k | s_n$ for $1 \le k \le m_n$. The set

$$A_{n+1} = \{a_k s_n : 1 \le k \le m_n - 1\} \cup \{a_k a_n : 1 \le k \le m_n\}$$

has $m_{n+1} = 2m_n - 1$ elements, each element is odd, and the sum of the elements of the set is $s_{n+1} = s_n^2$. It follows that each element of A_{n+1} is a divisor of s_{n+1}. Because $m_1 = 9 > 1$, it follows that $\{m_n\}$ is a strictly increasing sequence of integers, each with the desired property.

Note: Many readers noted that each of the integers s_n must be an odd perfect number or an odd abundant number. (Of course it is still an open question as to whether or not there exists an odd perfect number.) Readers also noted the relationship to the problem of writing 1 as a sum of distinct unit fractions with odd denominators,

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_m} = 1.$$

Indeed, given a solution in distinct odd integers $a_1, \ldots, a_m$, let L be the least common multiple of $a_1, \ldots, a_m$. Then $\{L/a_1, L/a_2, \ldots, L/a_m\}$ is a set whose elements sum to L, and each element of the set is a divisor of L.

Solution by JPV Abad, Alfred University Number Theory Group, Brian D. Beasley, J. C. Binz (Switzerland), John Christopher, Con Amore Problem Group (Denmark), Daniele Donini (Italy), Robert L. Doucette, Fejentalaltuka Szeged Problem Solving Semigroup (Hungary), Marty Getz and Dixon Jones, Mark Kidwell, Murray S. Klamkin (Canada), Ken Korbin, Victor Y. Kutsenok, Kathleen E. Lewis, Reiner Martin, Allen J. Mauney, Kevin McDougal, Scott Parker, Michael Reid, Jeremy Rouse, Phillip D. Straffin, Alexey Vorobyov, Li Zhou, and the proposer. There was one incorrect submission.

Answers

Solutions to the Quickies from page 318.

A923. Applying Hölder's Inequality for the particular value of m we find

$$\begin{aligned} 1 = \sum_{k=1}^{n} x_k^r &= \sum_{k=1}^{n} \frac{x_k^r}{y_k^{sm/(m+1)}} y_k^{sm/(m+1)} \\ &\le \left(\sum_{k=1}^{n} \frac{x_k^{r(m+1)}}{y_k^{sm}} \right)^{1/(m+1)} \left(\sum_{k=1}^{n} y_k^s \right)^{m/(m+1)} = 1. \end{aligned}$$

Because we have equality and $\sum x_k^r = \sum y_k^s$, it follows that $x_k^r = y_k^s$, $1 \le k \le n$. The desired result follows immediately.

A924. Let $\mathbf{e}_1 = \langle 1, \cos\beta, \cos\alpha, 1\rangle$, $\mathbf{e}_2 = \langle 1, 1, \cos\gamma, \cos\beta\rangle$, $\mathbf{e}_3 = \langle 1, \cos\gamma, 1, \cos\alpha\rangle$, and $\mathbf{e}_4 = \langle 1, 1, 1, 1\rangle$ be the edges of a parallelepiped in $\mathbb{R}^4$. Given the edge lengths for a parallelepiped, the volume is maximized when the edges are orthogonal. Hence,

$$\|\mathbf{e}_1\|^2 \|\mathbf{e}_2\|^2 \|\mathbf{e}_3\|^2 \|\mathbf{e}_4\|^2 \ge \det(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)^2. \qquad (*)$$

The expression on the right of $(*)$ expands to

$$256\sin^4\frac{\alpha}{2}\sin^4\frac{\beta}{2}\sin^4\frac{\gamma}{2},$$

while the expression on the left is equal to

$$4(2+\cos^2\alpha+\cos^2\beta)(2+\cos^2\beta+\cos^2\gamma)(2+\cos^2\gamma+\cos^2\alpha).$$

This establishes the result.

Next, notice that the four vectors are orthogonal (and equality holds in $(*)$) if and only if $\cos\alpha=\cos\beta=\cos\gamma=-1$, that is, when each of α, β, and γ is an odd multiple of π.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles and books are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Agrawal, Manindra, Neeraj Kayal, and Nitin Saxena, PRIMES is in P, `http://www.cse.iitk.ac.in/news/primality.html` . Robinson, Sara, New method said to solve key problem in math, *New York Times* (8 August 2002) A20; `http://www.nytimes.com/2002/08/08/science/08MATH.html` . Cipra, Barry, Simple recipe creates acid test for primes, *Science* 297 (16 August 2002) 1105–1106. Caldwell, Chris, The Prime Pages (prime number research, records and resources), `http://primes.utm.edu/` .

A new result from a professor and two undergraduates at the Indian Institute of Technology Kanpur is that whether an integer n is prime can be determined in polynomial time in its length—in particular, in $\mathcal{O}\big((\log n)^{12} q(\log\log n)\big)$ time for some polynomial q. (The exponent is 6 if a conjecture is true about the density of primes p for which $2p+1$ is also prime). The new deterministic algorithm is not faster than randomized primality tests, but its existence may spur mathematicians to discover faster deterministic algorithms. The algorithm itself is very short; and the proofs of its correctness and of its running time are straightforward, depending only on two lemmas from analytic number theory.

Peterson, Ivars, Prime effort: Powerful conjecture may be proved, *Science News* 161 (25 May 2002) 324–325; updated as Math Trek: Conquering Catalan's conjecture (24 June 2002) `http://www.maa.org/mathland/mathtrek_06_24_02.html`. Primary cyclotomic units and a proof of Catalan's conjecture (draft), Mihailescu, Preda, `http://www-math.uni-paderborn.de/~preda/papers/catcrelle.ps` . Fasel, Andreas, Geniestreich eines spät Berufenen [Stroke of genius from a late bloomer], *Die Welt* (23 June 2002) `http://www.welt.de/daten/2002/06/23/0623vm340077.htx` .

Eugène Charles Catalan conjectured in 1844 that the only solution to $x^m - y^n = 1$ in integers $x, y, m, n \geq 2$ is $3^2 - 2^3 = 1$. Preda Mihailescu (University of Paderborn, Germany), who at age 42 received his doctorate in 1997, appears to have proved the conjecture at last. Surprisingly, it is the newspaper account in *Die Welt* that gives more mathematical details, comparing the proofs of Catalan's conjecture and Fermat's Last Theorem: For both, the main approach for a hundred years was through Kummer theory, and both problems can lead to Frey elliptic curves. Elliptic curves were the key for FLT but a dead end for Catalan's conjecture; instead, Mihailescu succeeded by going back to Kummer theory.

Ascher, Martha, *Mathematics Elsewhere: An Exploration of Ideas Across Cultures*, Princeton University Press, 2002; x + 207 pp, \$24.95. ISBN 0–691–07020–2.

Continuing the theme of her earlier *Ethnomathematics: A Multicultural View of Mathematical Ideas* (Brooks/Cole, 1991), Ascher here presents more examples of mathematical ideas of traditional cultures throughout the world. This volume features the logic of divination (Madagascar), calendar-making (Trobriand Islands, Bali, and the Maya), map-making (Oceania), social relations (the Basques, Tonga, and the Borana of East Africa), and the kolam figures of Tamils in southern India.

2002 Fields Medals and Nevanlinna Prize Awarded, http://www.maa.org/news/fields02.html . Lanfranco, Ed, Top prizes in math awarded in China, UPI, (21 August 2002), http://www.upi.com/view.cfm?StoryID=20020821-105818-3022r . Nutt, Amy Ellis, His study of shapes garners top medal, (Newark, NJ) *Star-Ledger* (20 August 2002) 1, 14; http://www.njo.com/news/ledger/index.ssf%3F/base/news-3/102983462228939.xml . Peterson, I., Math prizes: Honors for connecting number theory, geometry, and algebra, *Science News* 162 (8) (24 August 2002) 117. Riehm, Carl, The early history of the Fields Medal, *Notices of the American Mathematical Society* 49 (7) (August 2002) 778–782.

Laurent Lafforgue (Institut des Hautes Études Scientifiques, Bures-sur-Yvette, France) and Vladimir Voevodsky (Institute for Advanced Study, Princeton) were awarded Fields Medals at the International Congress of Mathematicians in Beijing in late August. Lafforgue was recognized for proof of the Langlands conjecture for algebraic function fields (finite extensions of the field $\mathbb{C}(z)$ of rational functions in an indeterminate); this is part of the Langlands program of conjectures about connections between Galois representations and automorphic forms. Voevodsky has developed a new theory of cohomology that bridges number theory and algebraic geometry. Madhu Sudan (MIT) received the Nevanlinna Prize in mathematical computer science for work on probabilistically checkable proofs, non-approximability of certain NP-hard problems, and error-correcting codes. (But why did the Chinese President, Jiang Zemin, present the prizes?)

Seppa, N., Vaccine for All? Math model supports mass smallpox inoculation, *Science* 162 (13 July 2002) 21–22. Kaplan, Edward H., David L. Craft, and Lawrence M. Wein. Emergency response to a smallpox attack: The case for mass vaccination. *Proceedings of the National Academy of Sciences of the U.S.A.* 99 (2002) 10935–10940; http://www.pnas.org/cgi/content/full/99/16/10935 .

"Widespread vaccinations after a terrorist attack with the smallpox virus would save thousands more lives than the response plan currently being considered by the U.S. government." Edward H. Kaplan (Yale School of Management) and co-authors from the Operations Research Center and the Sloan School of Management at MIT disagree with the Centers for Disease Control (CDC) plan, which involves quarantining the sick and vaccinating their contacts. CDC officials declined comment on the study.

Hibbard, Allen C., and Ellen J. Maycock (eds.), *Innovations in Teaching Abstract Algebra*, MAA, 2002; x + 152 pp (P), $27.95 ($21.95 to members). ISBN 0–88385–171–7.

The MAA continues its sequence of books on resources for teaching specific courses with this one for abstract algebra. Five essays discuss "engaging" students, nine more discuss experiences with assorted software, and four are devoted to "learning algebra through applications and problem solving." Notable are three papers (all with software available) that respectively develop group-theoretical generalizations of colored Pascal triangles and hexagons, give a project based on the game "Lights Out," and teach permutation group theory via two-dimensional puzzles.

Malkevitch, Joseph, Pancakes, graphs, and the genome of plants, in *Proceedings of a Festschrift Held on the Occasion of the Retirement of Professor Robert Bumcrot*, edited by Steven R. Costenoble et al., 154–164; Dept. of Mathematics, Hofstra University, 2002.

Malkevitch starts from a 1975 problem in the *American Mathematical Monthly* about rearranging a stack of pancakes in order of size by repeatedly "grabbing several from the top and flipping them over." For n pancakes, how many flips may be needed? Malkevitch generalizes the operation, considers various questions, points out a graph-theoretic interpretation, and suggests projects (for high-school or college students). He also notes applications to computer architecture and to molecular biology and cites a paper in *Discrete Mathematics* by Bill Gates.

NEWS AND LETTERS

Carl B. Allendoerfer Awards — 2002

The Carl B. Allendoerfer Awards, established in 1976, are made to authors of expository articles published in *Mathematics Magazine*. The Awards are named for Carl B. Allendoerfer, a distinguished mathematician at the University of Washington and President of the Mathematical Association of America, 1959–60.

Mark McKinzie and Curtis Tuckey, Higher Trigonometry, Hyperreal Numbers, and Euler's Analysis, this MAGAZINE **74**:5 (December 2001), 339–368.

In his 1748 precalculus textbook *Introduction in Analysin Infinitorum,* Euler included the usual infinite-series expansions of trigonometric functions and other aspects of the "higher trigonometry." The authors begin this compelling article with a provocative discussion centered about Euler's claim that "All this follows from ordinary algebra" (including the arithmetic of infinite and infinitesimal quantities). They note that this has given rise to Euler's frequent portrayal as a reckless symbol-manipulator who worked in a number system fraught with nonsense and contradiction, but who through sheer intuitive brilliance came to correct conclusions.

They describe a modern context, replete with infinite and infinitesimal numbers, in which Euler's methods can be made intelligible, rigorous, and useful to modern readers—namely, the system of hyperreal numbers as described in Keisler's textbook *Calculus: An Infinitesimal Approach*. In a *tour de force* of expository clarity, they describe this system and use it to show how a large number of Euler's formal manipulations can be turned into rigorous proofs.

The article ends by saying, "Whether or not our rehabilitation of Euler's methods finds its way into the educational mainstream, we hope that by focusing our attention on the intellectual beauty of the underlying mathematics, we have convinced the reader that Euler's insights and arguments, far from being reckless or nonsensical, are directly relevant to the understanding, appreciation, and application of mathematics even in our own day."

The reader who is interested in Euler's methods, in extensions of the real number system, or in alternate approaches to some of the most important parts of calculus—that reader will enjoy this article.

Biographical Notes Curtis Tuckey is director of the Voice Laboratory, Oracle Corporation. Before joining Oracle he held various research and development positions at Motorola, Lucent Technologies, AT&T, and General Motors. He has occasionally taught at Loyola, DePaul, and Northwestern Universities. He has a Ph.D. from the University of Wisconsin, where he wrote a dissertation in nonstandard analysis under H. J. Keisler, and has a B.S. from Michigan State University, where he was first introduced to nonstandard analysis in a logic class taught by Jacob Plotkin. This paper was written while he was a research member of the Information Sciences Division of Bell Laboratories.

Born and raised in California, Mark McKinzie completed his undergraduate work in mathematics at San Jose State University. It was at SJSU, in one of Mack Stanley's logic classes, that he had his first exposure to nonstandard analysis. Mark's doctoral dissertation, on the early history of power series, was completed at the University of Wisconsin under the supervision of Michael Bleicher. Mark's current mathematical interests are the history of mathematics and the mathematical preparation of elementary teachers.

Since fall 1999, Mark has been a member of the Mathematics Department of Monroe Community College in Rochester, NY, where he shares a home with his wife Heather Ames Lewis (a mathematics professor at Nazareth College), their two-year-old son Emmett, and two cats (Leonhard Euler and Maria Agnesi).

Response from McKinzie/Tuckey Read Euler, he is the "master of us all." Laplace's injunction has been widely quoted, and with John Blanton's recent translations of several of Euler's monographs into English, Euler's original work is attracting renewed interest. His *Introductio in Analysin Infinitorum* (Introduction to the Analysis of Infinities) would seem to be a natural place to start one's reading: in the words of Carl Boyer, it is "the foremost textbook of modern times," and as a precalculus textbook it lays the foundation for Euler's later textbooks on the differential and integral calculus.

The stalwart reader of the *Introductio* is rewarded with a self-contained treatment of infinite series (including the development of transcendental functions into power series), infinite products (including the product representation for the sine, and consequently the celebrated formula $\sum 1/n^2 = \pi^2/6$), and a treatment of the foundations of the exponential and logarithmic functions, among many other gems.

But there is a barrier that discourages most modern readers from making headway, and that is Euler's explicit and essential use of infinite and infinitesimal numbers. We hope that our article will help make Euler's careful exposition more accessible to our contemporaries, by showing how one can sensitively rehabilitate Euler's arguments and calculations in a modern context without significantly altering their original sense and form.

We immensely enjoyed writing this article, and we are delighted that it has been selected for a Carl B. Allendoerfer Award. Our special thanks go to Richard Askey, Michael Benedikt, John Blanton, Michael Bleicher, Frank Farris, Dan Kalman, and Jerry Keisler, for sharing many insights on Euler's "analysis of infinities."

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